# DERIVED CATEGORIES WITH APPLICATIONS TO REPRESENTATIONS OF ALGEBRAS

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## Contents



## 1. Categories and Functors

**Definition 1.1** (Category)**.** We define a *category* $\mathcal{C}$ by the following data:

1. A class $\mathrm{Ob}\mathcal{C}$ of elements called objects of $\mathcal{C}$.
2. For a ordered pair $(X, Y)$ of objects a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms is given such that $\mathrm{Hom}_{\mathcal{C}}(X, Y) \cap \mathrm{Hom}_{\mathcal{C}}(X', Y') = \phi$ for $(X, Y) \neq (X', Y')$ (an element $f$ of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is called a morphism, and denote by $f : X \to Y$).

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3. For each triple $(X, Y, Z)$ of objects of $\mathcal{C}$ a map
$$\theta(X,Y,Z) : \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)$$
($\theta$ is called the composition map) is given.
4. The composition map $\theta$ is associative.
5. For each object $X$ of $\mathcal{C}$, there is a morphism $1_X : X \to X$ such that for any $g : Y \to X$, $h : X \to Z$ we have $1_X g = g, h1_X = h$.

$X \in \mathrm{Ob}\,\mathcal{C}$ (often $X \in \mathcal{C}$) means that $X$ is an object of $\mathcal{C}$.

**Example 1.2.** The following often appear in this note.

1. $\mathfrak{Set}$ is the category consisting of sets as objects and maps as morphisms.
2. $\mathfrak{Ab}$ is the category consisting of abelian groups as objects and group morphisms as morphisms.
3. For a ring $A$, $\mathsf{Mod}\,A$ is the category consisting of right $A$-modules as objects and $A$-homomorphisms as morphisms.

**Definition 1.3** (Opposite Category)**.** For a category $\mathcal{C}$, the *opposite category* $\mathcal{C}^{\mathrm{op}}$ of $\mathcal{C}$ is defined by

1. $\mathrm{Ob}\,\mathcal{C}^{\mathrm{op}} = \mathrm{Ob}\,\mathcal{C}$.
(for $X \in \mathcal{C}$, we denote by $X^{\mathrm{op}} \in \mathcal{C}^{\mathrm{op}}$ the same object)
2. For $X^{\mathrm{op}}, Y^{\mathrm{op}} \in \mathrm{Ob}\,\mathcal{C}^{\mathrm{op}}$,
$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X^{\mathrm{op}}, Y^{\mathrm{op}}) = \mathrm{Hom}_{\mathcal{C}}(Y, X).$$
(for $f \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$, we denote by $f^{\mathrm{op}} \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X^{\mathrm{op}}, Y^{\mathrm{op}})$)
3. The composition map $\theta^{\mathrm{op}}$ is defined by $\theta^{\mathrm{op}}(f^{\mathrm{op}}, g^{\mathrm{op}}) = \theta(g,f)^{\mathrm{op}}$.

**Definition 1.4.** Let $f : X \to Y$ be a morphism in a category $\mathcal{C}$.

1. $f$ is called a *monomorphism* if $fu = fv$ implies $u = v$.
2. $f$ is called an *epimorphism* if $uf = vf$ implies $u = v$.
3. $f$ is called a *split monomorphism* if there is $g : Y \to X$ such that $gf = 1_X$.
4. $f$ is called a *split epimorphism* if there is $g : Y \to X$ such that $fg = 1_Y$.
5. $f$ is called an *isomorphism* if there is $g : Y \to X$ such that $gf = 1_X$ and $fg = 1_Y$.

We often write $\twoheadrightarrow$ for an epimorphism, and $\rightarrowtail$ for a monomorphism.

**Definition 1.5** (Functor)**.** For categories $\mathcal{C}$ and $\mathcal{C}'$, a *covariant functor* (resp., *contravariant functor*) $F : \mathcal{C} \to \mathcal{C}'$ consists of the following data:

1. A map $F : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{C}'$.
2. For $X, Y \in \mathrm{Ob}\,\mathcal{C}$, a map
$$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}'}(FX, FY)$$
$$(\text{resp., } F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}'}(FY, FX))$$
such that $F(gf) = F(g)F(f)$ (resp., $F(gf) = F(f)F(g)$), $F(1_X) = 1_{F(X)}$.

Here we write simply $F(f)$ instead of $F_{X,Y}(f)$.

**Example 1.6.** In a category $\mathcal{C}$, for $X \in \mathcal{C}$, we define the covariant (resp., contravariant) functor
$$h^X : \mathcal{C} \to \mathfrak{Set}$$
$$(\text{resp., } h_X : \mathcal{C} \to \mathfrak{Set})$$
by $h^X(Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$ (resp., $h_X(Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X)$).

**Definition 1.7** (Functorial Morphism)**.** For covariant (resp., contravariant) functors $F, G : \mathcal{C} \to \mathcal{C}'$, a *functorial morphism* $\alpha : F \to G$ consists of the following data:

1. For each $X \in \mathcal{C}$, $\alpha_X : FX \to GX$ in $\mathcal{C}'$ is given.
2. For any morphism $f : X \to Y$ (resp., $f : Y \to X$) in $\mathcal{C}$, we have the following commutative diagram in $\mathcal{C}'$

$$\begin{array}{ccc} FX & \xrightarrow{\alpha_X} & GX \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ FY & \xrightarrow{\alpha_Y} & GY. \end{array}$$

In the case that a functorial morphism $\alpha$ is called a *functorial isomorphism* if $\alpha_X$ are isomorphisms for all $X \in \mathcal{C}$.

We denote by $\mathrm{Mor}(F, G)$ the collection of all functorial morphisms from $F$ to $G$.

**Lemma 1.8** (Representable Functor)**.** *For a covariant (resp., contravariant) functor $F : \mathcal{C} \to \mathfrak{Set}$, the following are equivalent for $C \in \mathcal{C}$.*

1. *$F$ is isomorphic to $h^C$ (resp., $h_C$).*
2. *There exists $c \in F(C)$ satisfying that for any $X \in \mathcal{C}$ and $x \in F(X)$, there is a unique $f \in \mathrm{Hom}_{\mathcal{C}}(C, X)$ (resp., $f \in \mathrm{Hom}_{\mathcal{C}}(X, C)$) such that $x = F(f)(c)$.*

*A covariant (resp., contravariant) functor $F : \mathcal{C} \to \mathfrak{Set}$ is called representable if there exists $C \in \mathcal{C}$ such that $F$ is isomorphic to $h^C$. Similarly, a contravariant functor $F' : \mathcal{C} \to \mathfrak{Set}$ is called representable if there exists $C \in \mathcal{C}$ such that $F$ is isomorphic to $h_C$.*

**Lemma 1.9** (Yoneda's Lemma)**.** *For $X \in \mathcal{C}$ and a covariant (resp., contravariant) functor $F : \mathcal{C} \to \mathfrak{Set}$, we have the bijection*

$$\theta_- : FX \to \mathrm{Mor}(h^X, F) \quad (\textit{resp., } \theta_- : FX \to \mathrm{Mor}(h_X, F)),$$

*where $\theta_-$ is defined by $(\theta_x)_Y(f) = F(f)(x)$ for $x \in FX$, $Y \in \mathcal{C}$, $f \in h^X(Y)$ (resp., $f \in h_X(Y)$).*

**Corollary 1.10.** *For $X, Y \in \mathcal{C}$, we have the bijection*

$$h^- : \mathrm{Hom}_{\mathcal{C}}(Y, X) \to \mathrm{Mor}(h^X, h^Y) \quad (\textit{resp., } h_- : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Mor}(h_X, h_Y)).$$

**Definition 1.11.** Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor.

1. $F$ is called *full* if $F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(FX, FY)$ are surjective for all $X, Y \in \mathcal{C}$.
2. $F$ is called *faithful* if $F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(FX, FY)$ are injective for all $X, Y \in \mathcal{C}$.
3. $F$ is called *dense* if for any $Y \in \mathcal{C}'$, there is $X \in \mathcal{C}$ such that $Y$ is isomorphic to $FX$.

**Definition 1.12** (Limit, Colimit)**.** Let $\mathcal{I}$, $\mathcal{C}$ be categories and $X \in \mathcal{C}$. We denote by $X_{\mathcal{I}} : \mathcal{I} \to \mathcal{C}$ the constant functor such that $X_{\mathcal{I}}(i) = X$ for all $i \in \mathcal{I}$ and $X_{\mathcal{I}}(f) = 1_X$ for all $f \in \mathrm{Hom}_{\mathcal{I}}(i, j)$.
For a functor $F : \mathcal{I} \to \mathcal{C}$, an object $X$ of $\mathcal{C}$ is called the *colimit* $\mathsf{colim}\ F$ (resp., the *limit* $\mathsf{lim}\ F$) of $F$ provided that for all $Y \in \mathcal{C}$ we have

$$\mathrm{Mor}(F, Y_{\mathcal{I}}) \cong \mathrm{Hom}_{\mathcal{C}}(X, Y)$$
$$(\text{resp.}, \mathrm{Mor}(Y_{\mathcal{I}}, F) \cong \mathrm{Hom}_{\mathcal{C}}(Y, X)).$$

**Definition 1.13** (Filtered Colimit)**.** A small category $\mathcal{I}$ is called a filtered category provided that

1. For any $i, j \in \mathcal{I}$, there exists $k \in \mathcal{I}$ and morphisms $i \to k$, $j \to k$ in $\mathcal{I}$.
2. For two morphisms $f, g : i \to j$, there is a morphism $h : j \to k$ such that $hf = hg$.

For a covariant (resp., contravariant) functor $F : \mathcal{I} \to \mathcal{C}$ from a filtered category $\mathcal{I}$ to a category $\mathcal{C}$, the filtered colimit $\varinjlim F$ (resp., the filtered limit $\varprojlim F$) of $F$ is the colimit $\mathsf{colim}\ F$ (resp., the limit $\mathsf{lim}\ F$).

**Definition 1.14** (Product, Coproduct)**.** For a collection $\{X_i\}_{i\in I}$ of objects indexed by a set $I$, $X$ is called a *coproduct* $\coprod_{i\in I} X_i$ (resp., a *product* $\prod_{i\in I} X_i$) of $\{X_i\}_{i\in I}$ provided that

1. There are a collection of morphisms $\{q_i : X_i \to X\}_{i\in I}$ (resp., $\{p_i : X \to X_i\}_{i\in I}$).
2. For any $Y \in \mathcal{C}$ and $\{f_i : X_i \to Y\}_{i\in I}$ (resp., $\{p_i : Y \to X_i\}_{i\in I}$), there exists a unique morphism $f : X \to Y$ (resp., $f : Y \to X$) with $f_i = fq_i$ (resp., $f_i = p_i f$) for all $i$.

If a coproduct $\coprod_{i\in I} X_i$ is also a product, then it is called a biproduct of $\{X_i\}_{i\in I}$ and denoted by $\bigoplus_{i\in I} X_i$.

**Definition 1.15** (Bifunctor)**.** Let $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{D}$ be categories. The *product category* $\mathcal{C}_1 \times \mathcal{C}_2$ is the category consisting of pairs $(X_1, X_2)$ of objects $X_1 \in \operatorname{Ob}\mathcal{C}_1$ and $X_2 \in \operatorname{Ob}\mathcal{C}_2$ as objects, and pairs $(f_1, f_2)$ of morphisms $f_1$ in $\mathcal{C}_1$ and $f_2$ in $\mathcal{C}_2$ as morphisms. A *bifunctor* is the functor $F : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$.

For bifunctors $F, G : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$, a *bifunctorial morphism* $\alpha : F \to G$ is a functorial morphism of functors $\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$.

Then a bifunctor $F : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$ consists of the following data:

1. For $X_1 \in \mathcal{C}_1$, $F(X_1, -) : \mathcal{C}_2 \to \mathcal{D}$ is a functor.
2. For $X_2 \in \mathcal{C}_2$, $F(-, X_2) : \mathcal{C}_1 \to \mathcal{D}$ is a functor.
3. For a morphism $f_1 : X_1 \to Y_1$ in $\mathcal{C}_1$, $F(f_1, -) : F(X_1, -) \to F(Y_1, -)$ is a functorial morphism.
   (or equivalently, for a morphism $f_2 : X_2 \to Y_2$ in $\mathcal{C}_1$, $F(-, f_2) : F(-, X_2) \to F(-, X_2)$ is a functorial morphism.)

And a bifunctorial morphism $\alpha : F \to G$ consists of the following data:

1. For each $(X_1, X_2) \in \mathcal{C}_1 \times \mathcal{C}_2$, $\alpha_{(X_1,X_2)} : F(X_1, X_2) \to G(X_1, X_2)$ in $\mathcal{D}$ is given.
2. For $X_1 \in \mathcal{C}_1$, $\alpha_{(X_1,-)} : F(X_1, -) \to G(X_1, -)$ is a functorial morphism.
3. For $X_2 \in \mathcal{C}_2$, $\alpha_{(-,X_2)} : F(-, X_2) \to G(-, X_2)$ is a functorial morphism.

In the case that a bifunctorial morphism $\alpha$ is called a *bifunctorial isomorphism* if $\alpha_{(X_1,X_2)}$ are isomorphisms for all $(X_1, X_2) \in \mathcal{C}_1 \times \mathcal{C}_2$.

**Definition 1.16** (Adjoint)**.** Let $F : \mathcal{C} \to \mathcal{C}'$, $G : \mathcal{C}' \to \mathcal{C}$ be covariant functors. We say that $F$ is a *left adjoint* of $G$ (or $G$ is a *right adjoint* of $F$) (denote by $F \dashv G$) if there is a bifunctorial isomorphism $t(-, ?) : \operatorname{Hom}_{\mathcal{C}'}(F-, ?) \to \operatorname{Hom}_{\mathcal{C}}(-, G?)$.
In this case, let $\sigma_X = t(X, FX)(1_{FX})$ and $\tau_Y = t(GY, Y)^{-1}(1_{GY})$ for $X \in \mathcal{C}$, $Y \in \mathcal{C}'$. ($\sigma : \mathbf{1}_{\mathcal{C}} \to GF$ and $\tau : FG \to \mathbf{1}_{\mathcal{C}'}$ are called the *adjunction arrows*.)

For contravariant functors $F' : \mathcal{C} \to \mathcal{C}'$, $G' : \mathcal{C}' \to \mathcal{C}$, a pair $(F', G')$ is called a *right adjoint pair* if there is a bifunctorial isomorphism $t'(-, ?) : \operatorname{Hom}_{\mathcal{C}'}(-, F?) \to \operatorname{Hom}_{\mathcal{C}}(?, G'-)$. There are adjunction arrows $\sigma : \mathbf{1}_{\mathcal{C}} \to G'F'$ and $\tau : \mathbf{1}_{\mathcal{C}'} \to F'G'$.

**Remark 1.17.** According to Corollary 1.10, it is easy to see that a right (resp., left) adjoint is uniquely determined up to isomorphism. And in the above, we have

$$G\tau \circ \sigma G = 1_G \text{ and } \tau F \circ F\sigma = 1_F.$$

**Theorem 1.18.** *For a covariant functor $F : \mathcal{C} \to \mathcal{C}'$, the following hold.*

1. *$F$ has a right adjoint if and only if $h_Y \circ F : \mathcal{C} \to \mathfrak{Set}$ is representable for any $Y \in \mathcal{C}'$.*
2. *$F$ has a left adjoint if and only if $h^X \circ F : \mathcal{C} \to \mathfrak{Set}$ is representable for any $X \in \mathcal{C}'$.*

**Theorem 1.19.** *Let $F : \mathcal{C} \to \mathcal{C}'$, $G : \mathcal{C}' \to \mathcal{C}$ be covariant functors such that $F \dashv G$. Then the following are equivalent.*

1. *$G$ is fully faithful.*
2. *The adjunction arrow $\tau : FG \to \mathbf{1}_{\mathcal{C}'}$ is a functorial isomorphism.*

*Sketch.*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X,Y) & \xrightarrow{\mathrm{Hom}(\tau_X, Y)} & \\ \downarrow G_{X,Y} & \searrow & \\ \mathrm{Hom}_{\mathcal{C}}(GX,GY) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{C}}(FGX,Y) \end{array}$$

□

**Theorem 1.20** (Equivalence)**.** *For a functor $F : \mathcal{C} \to \mathcal{C}'$, the following are equivalent.*

1. *$F$ is fully faithful and dense.*
2. *There is a functor $G : \mathcal{C}' \to \mathcal{C}$ such that $GF \cong \mathbf{1}_{\mathcal{C}}$ and $FG \cong \mathbf{1}_{\mathcal{C}'}$.*

*In this case, $F$ is called an equivalence and we say that $\mathcal{C}$ and $\mathcal{C}'$ are equivalent.*

**Theorem 1.21.** *Let $F : \mathcal{C} \to \mathcal{C}'$ be a covariant functor and let $G$ be a right adjoint of $F$. Then the following hold.*

1. *$F$ preserves the colimit in $\mathcal{C}$ of any functor.*
2. *$G$ preserves the limit in $\mathcal{C}'$ of any functor.*

## 2. Additive Categories and Abelian Categories

In a category $\mathcal{C}$, an object $U$ is called an *initial object* if for any $X \in \mathcal{C}$ $\mathrm{Hom}_{\mathcal{C}}(U,X)$ has only one element, $V$ is called a *terminal object* if for any $X \in \mathcal{C}$ $\mathrm{Hom}_{\mathcal{C}}(X,V)$ has only one element, and $O$ is called a *null object* if $O$ is initial and terminal.

**Definition 2.1** (Preadditive Category)**.** A category $\mathcal{C}$ with a null object is called a *preadditive category* provided that $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is an abelian group for any $X, Y \in \mathcal{C}$, and that the composition map $\theta$ is bilinear.

**Definition 2.2** (Additive functor)**.** Let $\mathcal{C}$, $\mathcal{C}'$ be preadditive categories. A covariant (resp., contravariant) functor $F : \mathcal{C} \to \mathcal{C}'$ between preadditive categories is called an *additive functor* provided that for $X, Y \in \mathcal{C}$,

$$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}'}(FX,FY)$$
$$(\text{resp., } F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}'}(FY,FX))$$

is a group morphism.

**Proposition 2.3.** *Let $\{X_i\}_{1\leq i\leq n}$ be a finite collection of objects of a preadditive category $\mathcal{C}$. Then the following are equivalent.*

1. *A coproduct $\coprod_{i=1}^n X_i$ of $\{X_i\}_{1\leq i\leq n}$ exists in $\mathcal{C}$.*
2. *A product $\prod_{i=1}^n X_i$ of $\{X_i\}_{1\leq i\leq n}$ exists in $\mathcal{C}$.*
3. *There exist an object $X \in \mathcal{C}$ and morphisms $u_i : X_i \to X$, $p_i : X \to X_i$ $(1 \leq i \leq n)$ such that*

$$(a) \qquad \Sigma_{i=1}^n u_i p_i = 1_X.$$

$$(b) \qquad p_i u_j = \begin{cases} 0 & \text{if } i \neq j \\ 1_{X_i} & \text{if } i = j. \end{cases}$$

*Moreover, the above coproduct is naturally isomorphic to the above product.*

**Proposition 2.4.** *Let $F : \mathcal{C} \to \mathcal{C}'$ be an additive functor between preadditive categories, $\{X_i\}_{1\leq i\leq n}$ a finite collection of objects in $\mathcal{C}$. If the coproduct $\coprod_{i=1}^n X_i$ exists in $\mathcal{C}$, then the coproduct $\coprod_{i=1}^n F(X_i)$ exists in $\mathcal{C}'$ and is canonically isomorphic to $F(\coprod_{i=1}^n X_i)$.*

**Definition 2.5** (Additive Category)**.** A preadditive category $\mathcal{C}$ is called an *additive functor* if $\mathcal{C}$ satisfies Proposition 2.3 for any finite collection of objects in $\mathcal{C}$.

**Example 2.6.** Let $\mathcal{C}$ be an additive category. For $M \in \mathcal{C}$, We define $\mathsf{Add}\, M$ (resp., $\mathsf{add}\, M$) the full subcategory of $\mathcal{C}$ consisting of objects which are direct summands of coproducts (resp., finite coproducts) of copies of $M$. Then $\mathsf{Add}\, M$ (resp., $\mathsf{add}\, M$) is an additive category.

**Proposition 2.7** (Compact Object)**.** *For an object $C$ of an additive category $\mathcal{C}$, the following are equivalent.*

1. *For any morphism $f : C \to \coprod_{i\in I} X_i$, there exists a factorization*

$$C \xrightarrow{f'} \coprod_{j\in F} X_j \xrightarrow{\mu_F} \coprod_{i\in I} X_i$$

   *where $F$ is a finite subset of $I$ and $\mu_F$ is the canonical inclusion.*
2. *For any morphism $f : C \to \coprod_{i\in I} X_i$, there exists a finite subset $F$ of $I$ such that $f = \sum_{j\in F} u_j p_j f$ where $u_j$ are the structural morphisms and $p_j$ are the canonical projections.*
3. *The functor $h^C : \mathcal{C} \to \mathfrak{Ab}$ preserves coproducts.*

*An object $C \in \mathcal{C}$ is called a compact object (often called a small object) of $\mathcal{C}$ if $C$ satisfies the above conditions.*

**Exercise 2.8.** Show that if a right $A$-module $C$ is finitely generated, then $C$ is a compact object in $\mathsf{Mod}\, A$.

**Corollary 2.9.** *Let $C$ be a compact object of an additive category $\mathcal{C}$, and $B = \mathrm{End}_{\mathcal{C}}(C)$. The following hold.*

1. *For any object $X \in \mathcal{C}$,we have isomorphisms*

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(C^{(I)}, X) &\xrightarrow{\sim} \mathrm{Hom}_B(\mathrm{Hom}_{\mathcal{C}}(C, C^{(I)}), \mathrm{Hom}_{\mathcal{C}}(C, X)) \\ &\xrightarrow{\sim} \mathrm{Hom}_B(\mathrm{Hom}_{\mathcal{C}}(C, C)^{(I)}, \mathrm{Hom}_{\mathcal{C}}(C, X))\end{aligned}$$

   *if a coproduct $C^{(I)}$ exists for a set $I$.*

2. *we have isomorphisms*

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(C^{(I)}, C^{(J)}) &\xrightarrow{\sim} \mathrm{Hom}_B(\mathrm{Hom}_{\mathcal{C}}(C, C^{(I)}), \mathrm{Hom}_{\mathcal{C}}(C, C^{(J)}))\\ &\xrightarrow{\sim} \mathrm{Hom}_B(\mathrm{Hom}_{\mathcal{C}}(C, C)^{(I)}, \mathrm{Hom}_{\mathcal{C}}(C, C)^{(J)})\end{aligned}$$

*if coproducts $C^{(I)}, C^{(J)}$ exist for sets $I, J$.*

**Proposition 2.10.** *Let $\mathcal{C}$ be an additive category, $\mathcal{C}'$ a preadditive category and $F : \mathcal{C} \to \mathcal{C}'$ a functor. If $F$ preserves finite coproducts, then $F$ is an additive functor.*

**Definition 2.11** (Special Morphisms)**.** Let $\mathcal{C}$ be a preadditive category. For $f : X \to Y$, $g : X \to Z$ and $h : W \to Y$, we define the following.

1. $(\mathrm{Cok}\, f, \mathrm{cok}\, f) = \mathsf{colim}\ (X \xrightarrow{f} Y, X \xrightarrow{0} Y)$.
2. $(\mathrm{Ker}\, f, \ker f) = \mathsf{lim}\ (X \xrightarrow{f} Y, X \xrightarrow{0} Y)$.
3. $(\mathrm{Im}\, f, \mathrm{im}\, f) = \mathrm{Ker}(Y \xrightarrow{0} \mathrm{Cok}\, f, \mathrm{cok}\, f)$.
4. $(\mathrm{Coim}\, f, \mathrm{im}\, f) = \mathrm{Cok}(\mathrm{Ker}\, f \xrightarrow{0} X, \ker f)$.
5. $\mathrm{PushOut}(f, g) = \mathsf{colim}\ (X \xrightarrow{f} Y, X \xrightarrow{g} Z)$.
6. $\mathrm{PullBack}(f, h) = \mathsf{lim}\ (X \xrightarrow{f} Y, W \xrightarrow{h} Y)$.

**Proposition 2.12.** *Let $\mathcal{C}$ be a preadditive category and let $f : X \to Y$ be a morphism in $\mathcal{C}$ such that there exist* $\mathrm{Ker}\, f$, $\mathrm{Cok}\, f$, $\mathrm{Coim}\, f$ *and* $\mathrm{Im}\, f$ *in $\mathcal{C}$. Then there exists a unique morphism $\overline{f} : \mathrm{Coim}\, f \to \mathrm{Im}\, f$ such that we have the following commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{coim}\, f}\downarrow & & \uparrow{\scriptstyle \mathrm{im}\, f} \\ \mathrm{Coim}\, f & \xrightarrow{\overline{f}} & \mathrm{Im}\, f \end{array}$$

**Definition 2.13** (Abelian Category)**.** An additive category $\mathcal{C}$ is called an *abelian category* provided that

1. For any morphism $f$, there exist $\mathrm{Ker}\, f$ and $\mathrm{Cok}\, f$ in $\mathcal{C}$.
2. For any morphism $f$, the above morphism $\overline{f}$ is an isomorphism.

**Definition 2.14.** In an abelian category $\mathcal{C}$, we consider the following sequence

$$\ldots \to X^{i-1} \xrightarrow{f^{i-1}} X^i \xrightarrow{f^i} X^{i+1} \to \ldots.$$

We say that the above sequence is *exact* at $X^i$ if $\mathrm{Ker}\, f^i = \mathrm{Im}\, f^{i-1}$. If the above sequence is exact at each $X^i$, then we say that the above sequence is exact.

In the rest of this section, we deal with internal properties of an abelian category $\mathcal{C}$.

**Proposition 2.15** (Snake Lemma)**.** *Suppose that the following diagram is commutative*

$$\begin{array}{ccccccc} & & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & O \\ & & {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle z} & & \\ O & \longrightarrow & X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & & \end{array}$$

*where all rows are exact. Then we have the following induced exact sequence*

$$\operatorname{Ker} x \to \operatorname{Ker} y \to \operatorname{Ker} z \to \operatorname{Cok} x \to \operatorname{Cok} y \to \operatorname{Cok} z.$$

*Moreover, $f$ (resp., $g'$) is monic (resp., epic) if and only if so is* $\operatorname{Ker} x \to \operatorname{Ker} y$ *(resp.,* $\operatorname{Cok} y \to \operatorname{Cok} z$*).*

**Proposition 2.16** (Five Lemma)**.** *Suppose that the following diagram is commutative*

$$\begin{array}{ccccccccc} X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & X_4 & \longrightarrow & X_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & X_4 & \longrightarrow & X_5 \end{array}$$

*where all rows are exact. Then the following hold.*

1. *If $f_1$ is epic, and $f_2$, $f_4$ are monic, then $f_3$ is monic.*
2. *If $f_5$ is monic, and $f_2$, $f_4$ are epic, then $f_3$ is epic.*
3. *If $f_1$ is epic, $f_5$ is monic, and $f_2$, $f_4$ are isomorphisms, then $f_3$ is an isomorphism.*

**Proposition 2.17** (Pull Back 1)**.** *Suppose that the following diagram is commutative*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle x}\downarrow & (A) & \downarrow{\scriptstyle y} \\ X' & \xrightarrow{f'} & Y' \end{array}$$

*Then the following hold.*

1. *The square $(A)$ is pull back if and only if $O \to X \xrightarrow{\left[\begin{smallmatrix} f \\ x \end{smallmatrix}\right]} Y \oplus X' \xrightarrow{[-y\ f']} Y'$ is exact.*
2. *The square $(A)$ is push out if and only if $X \xrightarrow{\left[\begin{smallmatrix} f \\ x \end{smallmatrix}\right]} Y \oplus X' \xrightarrow{[-y\ f']} Y' \to O$ is exact.*

**Proposition 2.18** (Pull Back 2)**.** *Suppose that the following diagram is commutative*

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow & (A) & \downarrow & (B) & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' \end{array}$$

*If the squares $(A)$ and $(B)$ are push out (resp., pull back), then so is the square $(A) + (B)$.*

**Proposition 2.19** (Pull Back 3)**.** *Suppose that the following diagram is pull back (resp., push out)*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ X' & \xrightarrow{f'} & Y' \end{array}$$

*Then the following hold.*

1. *If $f'$ (resp., $f$) is epic (resp., monic), then the above diagram is also push out (resp., pull back), and $f$ (resp., $f'$) is also epic (resp., monic).*
2. *The induced morphism* $\operatorname{Ker} f \to \operatorname{Ker} f'$ *is an isomorphism (resp., an epimorphism).*
3. *The induced morphism* $\operatorname{Cok} f \to \operatorname{Cok} f'$ *is a monomorphism (resp., an isomorphism).*

**Proposition 2.20** (Exact Sequence 1)**.** *Suppose that the following diagram is commutative*

$$\begin{array}{ccccccccc} O & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & O \\ & & \downarrow & & \downarrow & & \| & & \\ O & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & Z & \longrightarrow & O \end{array}$$

*where all rows are exact. Then the square*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & EX & \downarrow \\ X' & \longrightarrow & Y' \end{array}$$

*is pull back and push out (this is called an exact square).*

**Proposition 2.21** (Exact Sequence 2)**.** *Suppose that the following diagram is commutative*

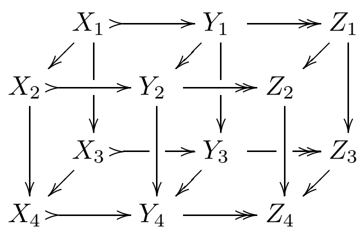
$$\begin{array}{ccccccccc} O & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & O \\ & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} & & \| & & \\ O & \longrightarrow & X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z & \longrightarrow & O \end{array}$$

*where all rows are exact. Then we have the following commutative diagram*

$$\begin{array}{ccccccccc} O & \longrightarrow & X & \xrightarrow{\alpha} & Y \oplus X' & \xrightarrow{\beta} & Y' & \longrightarrow & O \\ & & \| & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle g'} & & \\ O & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{-g} & Z & \longrightarrow & O \end{array}$$

*where* $\alpha = \left[\begin{smallmatrix} f \\ x \end{smallmatrix}\right]$, $\beta = \left[\begin{smallmatrix} -y & f' \end{smallmatrix}\right]$, $\gamma = \left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right]$, *where all rows are exact.*

**Proposition 2.22** (Exact Sequence 3)**.** *Suppose that the following diagram is commutative*



*where all rows are short exact sequences. If two of squares*

$$\begin{array}{ccccccccc} X_1 & \longrightarrow & X_2 & Y_1 & \longrightarrow & Y_2 & Z_1 & \longrightarrow & Z_2 \\ \downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow \\ X_3 & \longrightarrow & X_4 & Y_3 & \longrightarrow & Y_4 & Z_3 & \longrightarrow & Z_4 \end{array}$$

*are exact, then the rest is also exact.*

*Hint.* Consider the following commutative diagram

$$\begin{array}{ccccccccc} O & \longrightarrow & X_1 & \longrightarrow & Y_1 & \longrightarrow & Z_1 & \longrightarrow & O \\ & & \downarrow & & \downarrow & & \downarrow & & \\ O & \longrightarrow & X_1 \oplus X_2 & \longrightarrow & Y_3 \oplus Y_2 & \longrightarrow & Z_2 \oplus Z_1 & \longrightarrow & O \\ & & \downarrow & & \downarrow & & \downarrow & & \\ O & \longrightarrow & X_2 & \longrightarrow & Y_4 & \longrightarrow & Z_2 & \longrightarrow & O. \end{array}$$

□

**Exercise 2.23.** The following hold.

1. For a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$, if for any $M \in \mathcal{A}$,

$$0 \to \mathrm{Hom}_{\mathcal{A}}(Z, M) \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(g,M)} \mathrm{Hom}_{\mathcal{A}}(Y, M) \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(f,M)} \mathrm{Hom}_{\mathcal{A}}(X, M) \to 0$$

is exact, then

$$O \to X \xrightarrow{f} Y \xrightarrow{g} Z \to O$$

is split exact.

2. For a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$, if for any $M \in \mathcal{A}$,

$$0 \to \mathrm{Hom}_{\mathcal{A}}(M, X) \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(M,f)} \mathrm{Hom}_{\mathcal{A}}(M, Y) \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(M,g)} \mathrm{Hom}_{\mathcal{A}}(M, Z) \to 0$$

is exact, then

$$O \to X \xrightarrow{f} Y \xrightarrow{g} Z \to O$$

is split exact.

**Definition 2.24** (Ab$n$ Categories)**.** We define the conditions of an abelian category $\mathcal{C}$.

(Ab3) We say that an abelian category $\mathcal{C}$ satisfies the condition Ab3 (resp., Ab3*) if $\mathcal{C}$ has coproducts (resp., products) of objects indexed by arbitrary sets.

(Ab4) We say that an abelian category $\mathcal{C}$ satisfies the condition Ab4 (resp., Ab4*) provided that $\mathcal{C}$ satisfies the condition Ab3 (resp., Ab3*), and that the coproduct (resp., product) of monics (resp., epics) is monic (resp., epic).

(Ab5) We say that an abelian category $\mathcal{C}$ satisfies the condition Ab5 (resp., Ab5*) provided that $\mathcal{C}$ satisfies the condition Ab3 (resp., Ab3*), and that the filtered colimit (resp., filtered limit) of exact sequences is exact.

**Proposition 2.25.** *The following hold.*

1. *In a category satisfying Ab3* and Ab5, any* $\coprod \to \prod$ *is monic.*
2. *Ab5* $\Rightarrow$ *Ab4.*

3. *An abelian category $\mathcal{C}$ satisfies the condition Ab5 if and only if $\mathcal{C}$ satisfies the condition Ab3, and for a collection $\{X_i\}$ of subobjects of an object $X$, we have*
$$\sum\nolimits_i (X_i \cap X') = (\sum\nolimits_i X_i) \cap X'$$
*for any subobject $X'$ of $X$.*

**Example 2.26.** For a ring $A$, $\mathsf{Mod}\, A$ satisfies the conditions Ab4*, Ab5.

## 3. Krull-Schmidt Categories

Let $R$ be a ring with unity and let $\mathrm{J}(R)$ be the Jacobson radical of $R$. We call $R$ a semiperfect ring if (i) $R/\mathrm{J}(R)$ is a semi-simple Artinian ring, and (ii) any idempotent of $R/\mathrm{J}(R)$ can be lifted to an idempotent of $R$.

**Lemma 3.1** (Semiperfect Rings 1)**.** *The following hold.*

1. *A ring $R$ is semiperfect if and only if $R$ has a complete set of orthogonal primitive idempotents $e_i$ $(1 \le i \le n)$ such that each $e_iRe_i$ is a local ring.*
2. *A ring $R$ is semiperfect if and only if every finitely generated $R$-module has a projective cover.*

**Lemma 3.2** (Semiperfect Rings 2)**.** *Let $R$ be a semiperfect ring and let $e_i$ $(1 \le i \le n)$ be a complete set of orthogonal primitive idempotents.*

1. *If $f_i$ $(1 \le i \le m)$ is another complete set of orthogonal primitive idempotents, then $m = n$ and there is a permutation $\pi$ such that $Rf_i \cong Re_{\pi(i)}$ for all $i$.*
2. *If $f$ is an idempotent of $R$, then there are a permutation $\pi$ and an integer $t$ $(1 \le t \le n)$ such that $Rf \cong \bigoplus_{i=1}^{t} Re_{\pi(i)}$ and $R(1-f) \cong \bigoplus_{i=t+1}^{n} Re_{\pi(i)}$.*
3. *If $I$ is a two-sided ideal of $R$, then $R/I$ is also semiperfect.*

**Proposition 3.3.** *Let $\mathcal{C}$ be an additive category, and let $X \in \mathcal{C}$, $B = \mathrm{End}_{\mathcal{C}}(X)$. If $X'$ is a direct summand of a finite coproduct of copies of $X$, we have*
$$\mathrm{Hom}_{\mathcal{C}}(X', Y) \xrightarrow{\sim} \mathrm{Hom}_B(\mathrm{Hom}_{\mathcal{C}}(X, X'), \mathrm{Hom}_{\mathcal{C}}(X, Y)) \quad (f \mapsto \mathrm{Hom}_{\mathcal{C}}(X, f))$$
*for all $Y \in \mathcal{C}$.*

*Proof.* There are $q_i : X' \to X$ and $p_i : X \to X'$ $(1 \le i \le n)$ such that $\sum_{i=1}^{n} p_i q_i = 1_{X'}$. Let $\phi \in \mathrm{Hom}_B(\mathrm{Hom}_{\mathcal{C}}(X, X'), \mathrm{Hom}_{\mathcal{C}}(X, Y))$, for any $g \in \mathrm{Hom}_{\mathcal{C}}(X, X')$, we have
$$\begin{aligned}\phi(g) &= \phi\left(\sum\nolimits_{i=1}^{n} p_i q_i g\right)\\ &= \sum\nolimits_{i=1}^{n} \phi(p_i) q_i g\\ &= \mathrm{Hom}_{\mathcal{C}}\left(X, \sum\nolimits_{i=1}^{n} \phi(p_i) q_i\right)(g).\end{aligned}$$
Then $\mathrm{Hom}_{\mathcal{C}}(X, -)$ is surjective. Let $f \in \mathrm{Hom}_{\mathcal{C}}(X', Y)$ such that $\mathrm{Hom}_{\mathcal{C}}(X, f) = 0$. Then $fp_i = 0$ for all $i$, and hence $f = f\sum_{i=1}^{n} p_i q_i = \sum_{i=1}^{n} f p_i q_i = 0$. □

**Definition 3.4.** Let $\mathcal{C}$ be an additive category. An object $X$ of $\mathcal{C}$ is called *indecomposable* if $X \cong X_1 \oplus X_2$ implies $X_1 = O$ or $X_2 = O$.

**Definition 3.5** (Pre-Krull-Schmidt Category)**.** An additive category $\mathcal{C}$ is called a *pre-Krull-Schmidt category* provided that $\mathrm{End}_{\mathcal{C}}(X)$ is a semiperfect ring for each $X \in \mathcal{C}$.

**Proposition 3.6.** *Let $\mathcal{C}$ be a pre-Krull-Schmidt category. For any $X \in \mathcal{C}$, there are indecomposable objects $X_i$ $(1 \leq i \leq n)$ such that*

$$X \cong \bigoplus_{i=1}^{n} X_i.$$

*Proof.* Given $X \in \mathcal{C}$, since $\mathrm{End}_{\mathcal{C}}(X)$ is a semiperfect ring, there is a natural number $n_X$ such that $\mathrm{End}_{\mathcal{C}}(X)$ has a complete set of orthogonal primitive idempotents $e_i$ $(1 \leq i \leq n_X)$. If $X$ is not indecomposable, then we have a decomposition $X = X_1 \oplus X_2$ with $X_i \neq O$ $(i = 1, 2)$. By Lemma 3.2,2, Proposition 3.3, we have $n_{X_i} < n_X$ $(i = 1, 2)$. We get the statement by induction on $n_X$. □

**Proposition 3.7.** *Let $\mathcal{C}$ be a pre-Krull-Schmidt category. Then the following are equivalent.*

1. *For any object $X \in \mathcal{C}$, $X$ is indecomposable if and only if $\mathrm{End}_{\mathcal{C}}(X)$ is a local ring.*
2. *For any object $X \in \mathcal{C}$, for any $e^2 = e \in \mathrm{End}_{\mathcal{C}}(X)$ there exist $Y \in \mathcal{C}$ and $q : Y \to X$, $p : X \to Y$ such that $qp = e$ and $pq = 1_Y$ (i.e. any idempotent of $\mathrm{End}_{\mathcal{C}}(X)$ splits).*

*Proof.* $1 \Rightarrow 2$. For any object $X \in \mathcal{C}$, by Proposition 3.6, we have $X \cong \bigoplus_{i=1}^{n} X_i$, where $X_i$ are indecomposable objects $(1 \leq i \leq n)$. Then the compositions of natural morphisms $X \to X_i \to X$ form a complete set of orthogonal primitive idempotents of $\mathrm{End}_{\mathcal{C}}(X)$. By Lemma 3.2, 2, we get the statement 2.

$2 \Rightarrow 1$. Since $\mathrm{End}_{\mathcal{C}}(X)$ is semiperfect, it is trivial. □

**Definition 3.8** (Krull-Schmidt Category)**.** We call a pre-Krull-Schmidt category $\mathcal{C}$ a *Krull-Schmidt category* if $\mathcal{C}$ satisfies the equivalent conditions of Proposition 3.7.

**Theorem 3.9** (Krull-Schmidt Theorem)**.** *Let $\mathcal{C}$ be a Krull-Schmidt category. For any $X \in \mathcal{C}$, $X$ is isomorphic to $\bigoplus_{i=1}^{n} X_i$, where $X_i$ are indecomposable objects. Moreover, this decomposition is unique up to isomorphism (this is called a K-S decomposition).*

*Proof.* By Propositions 3.6, 3.7, $X \in \mathcal{C}$ has a K-S decomposition $\bigoplus_{i=1}^{n} X_i$. Lemma 3.2 and Proposition 3.3 imply uniqueness of this decomposition. □

**Example 3.10.** We denote by $\mathsf{mod}\, A$ the category of finitely presented right $A$-modules. Let $R$ be a commutative complete local ring, $A$ a finite $R$-algebra. Then $\mathsf{mod}\, A$ is a Krull-Schmidt category.

**Definition 3.11** (Stable Category)**.** Let $\mathcal{C}$ be an additive category, $\mathcal{I}$ an additive full subcategory of $\mathcal{C}$. For $X, Y \in \mathcal{C}$, let $\mathcal{I}(X, Y)$ be the subgroup of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ generated by morphisms which factor through some object of $\mathcal{I}$. We define the category $\underline{\mathcal{C}}_{\mathcal{I}}$ as follows.

1. $\mathrm{Ob}\, \underline{\mathcal{C}}_{\mathcal{I}} = \mathrm{Ob}\, \mathcal{C}$.
2. $\mathrm{Hom}_{\underline{\mathcal{C}}_{\mathcal{I}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)/\mathcal{I}(X, Y)$.

This category is called the *stable category* of $\mathcal{C}$ by $\mathcal{I}$.

**Remark 3.12.** For $X \in \mathcal{C}$, If $X$ is a direct summand of some object of $\mathcal{I}$, then $X \cong O$ in $\underline{\mathcal{C}}_{\mathcal{I}}$.

**Theorem 3.13.** *Let $\mathcal{C}$ be an additive category, $\mathcal{I}$ an additive full subcategory of $\mathcal{C}$. If $\mathcal{C}$ is a Krull-Schmidt category, then so is $\underline{\mathcal{C}}_{\mathcal{I}}$.*

*Proof.* By Proposition 2.3, $\underline{\mathcal{C}_{\mathcal{I}}}$ is clearly an additive category. For an indecomposable object $X \in \underline{\mathcal{C}_{\mathcal{I}}}$, we have a K-S-decomposition $X = \bigoplus_{i=1}^{n} X_i$ in $\mathcal{C}$. Let $e_i : X \xrightarrow{p_i} X_i \xrightarrow{q_i} X$ be the canonical morphism in $\mathcal{C}$ $(1 \le i \le n)$, and $\underline{e_i}$ be the image of $e_i$ in $\in \underline{\mathcal{C}_{\mathcal{I}}}$. If the number of $e_i$ such that $\underline{e_i} \neq 0$ is greater than 1, then by Lemma 3.2 this contradicts indecomposability of $X$. Thus we may assume that $\underline{e_1} \neq 0$ and $\underline{e_i} = 0$ for $i \ge 2$, and then $q_i p_i$ factors through $V_i \in \mathcal{I}$ for $i \ge 2$. Then $X_i$ is a direct summand of $V_i$. Therefore, by Lemma 3.2,3, $\mathrm{End}_{\underline{\mathcal{C}_{\mathcal{I}}}}(X) \cong \mathrm{End}_{\underline{\mathcal{C}_{\mathcal{I}}}}(X_1)$ is a local ring. We complete the proof by Proposition 3.7. □

**Example 3.14.** Let $R$ be a commutative complete local ring, $A$ a finite $R$-algebra, and $\mathsf{proj}\, A$ the full subcategory of $\mathsf{mod}\, A$ consisting of finitely generated projective right $A$-modules. Then the stable category $\underline{\mathsf{mod}} A$ of $\mathsf{mod}\, A$ by $\mathsf{proj}\, A$ is a Krull-Schmidt category.

## 4. Triangulated Categories

Throughout this section, unless otherwise stated, functors are covariant functors.

**Definition 4.1.** A *triangulated category* $\mathcal{C}$ is an additive category together with (1) an auto-equivalence $T : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$, called the *translation*, and (2) a collection $\mathcal{T}$ of sextuples $(X, Y, Z, u, v, w)$, called *triangle* (*distinguished triangle*) . These data are subject to the following four axioms:

(TR1) (1) Every sextuple $(X, Y, Z, u, v, w)$ which is isomorphic to a triangle is a triangle.
(2) Every morphism $u : X \to Y$ is embedded in a triangle $(X, Y, Z, u, v, w)$.
(3) The triangle $(X, X, O, 1_X, 0, 0)$ is a triangle for all $X \in \mathcal{C}$.

(TR2) A triangle $(X, Y, Z, u, v, w)$ is a triangle if and only if $(Y, Z, TX, v, w, -Tu)$ is a triangle.

(TR3) For any triangles $(X, Y, Z, u, v, w)$, $(X', Y', Z', u', v', w')$ and morphisms $f : X \to X'$, $g : Y \to Y'$ with $gu = u'f$, there exists $h : Z \to Z'$ such that $(f, g, h)$ is a homomorphism of triangles.

(TR4) (Octahedral axiom) For any two consecutive morphisms $u : X \to Y$ and $v : Y \to Z$, if we embed $u$, $vu$ and $v$ in triangles $(X, Y, Z', u, i, i')$, $(X, Z, Y', vu, k, k')$ and $(Y, Z, X', v, j, j')$, respectively, then there exist morphisms $f : Z' \to Y'$, $g : Y' \to X'$ such that the following diagram commute

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{i} & Z' & \xrightarrow{i'} & TX \\
\| & & \big\downarrow{\scriptstyle v} & & \big\downarrow{\scriptstyle f} & & \| \\
X & \xrightarrow{vu} & Z & \xrightarrow{k} & Y' & \xrightarrow{k'} & TX \\
 & & \big\downarrow{\scriptstyle j} & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle Tu} \\
 & & X' & = & X' & \xrightarrow{j'} & TY \\
 & & \big\downarrow{\scriptstyle j'} & & \big\downarrow{\scriptstyle (Ti)j'} & & \\
 & & TY & \xrightarrow{Ti} & TZ' & &
\end{array}$$

and the third column is a triangle.

Sometimes, we write $X[i]$ for $T^i(X)$.

**Definition 4.2** ($\partial$-functor)**.** Let $\mathcal{C}, \mathcal{C}'$ be triangulated categories. An additive functor $F : \mathcal{C} \to \mathcal{C}'$ is called *$\partial$-functor* (sometimes *exact functor*) provided that there is a functorial isomorphism $\alpha : FT_{\mathcal{C}} \xrightarrow{\sim} T_{\mathcal{C}'}F$ such that $(FX, FY, FZ, F(u), F(v), \alpha_X F(w))$ is a triangle in $\mathcal{C}'$ whenever $(X, Y, Z, u, v, w)$is a triangle in $\mathcal{C}$. Moreover, if a $\partial$-functor $F$ is an equivalence, then we say that $\mathcal{C}$ is *triangle equivalent* to $\mathcal{C}'$, and denote by $\mathcal{C} \overset{t}{\cong} \mathcal{C}'$

For $(F, \alpha), (G, \beta) : \mathcal{C} \to \mathcal{C}'$ $\partial$-functors, a functorial morphism $\phi : F \to G$ is called a *$\partial$-functorial morphism* if $(T_{\mathcal{C}'}\phi)\alpha = \beta\phi T_{\mathcal{C}}$.

We denote by $\partial(\mathcal{C}, \mathcal{C}')$ the collection of all $\partial$-functors from $\mathcal{C}$ to $\mathcal{C}'$, and denote by $\partial\,\mathrm{Mor}(F, G)$ the collection of $\partial$-functorial morphisms from $F$ to $G$.

**Definition 4.3.** Given a triangulated category $\mathcal{C}$ with a translation $T_{\mathcal{C}}$, we define the opposite triangulated category $\mathcal{C}^{\mathrm{op}}$ the following

1. $T_{\mathcal{C}^{\mathrm{op}}}(X^{\mathrm{op}}) = T_{\mathcal{C}}^{-1}(X)$.

2. $X^{\mathrm{op}} \to Y^{\mathrm{op}} \to Z^{\mathrm{op}} \to T_{\mathcal{C}^{\mathrm{op}}} X^{\mathrm{op}}$ is a distinguished triangle if $T_{\mathcal{C}}^{-1} X \to Z \to Y \to X$ is a distinguished triangle in $\mathcal{C}$.

**Definition 4.4.** A covariant additive functor $H : \mathcal{C} \to \mathcal{C}'$ from a triangulated category to an abelian category is called a *covariant cohomological functor*, if whenever (X, Y, Z, u, v, w) is a triangle in $\mathcal{C}$, the long sequence

$$\ldots \to H(T^i(X)) \xrightarrow{H(T^i(u))} H(T^i(Y)) \xrightarrow{H(T^i(v))} H(T^i(Z)) \xrightarrow{H(T^i(w))} H(T^{i+1}(X)) \to \ldots$$

is exact. If $H$ is a cohomological functor, then we often write $H^i(X)$ for $H(T^i(X))$, $i \in \mathbb{Z}$. One defines a *contravariant cohomological functor* by reversing the arrows.

In this section, we deal with internal properties of a triangulated category $\mathcal{C}$.

**Proposition 4.5.** *The following hold.*

1. *If $(X, Y, Z, u, v, w)$ is a triangle, then $vu = 0$, $wv = 0$ and $T(u)w = 0$.*
2. *For any $W \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(W, -) : \mathcal{C} \to \mathfrak{Ab}$ (resp., $\mathrm{Hom}_{\mathcal{C}}(-, W) : \mathcal{C} \to \mathfrak{Ab}$) is a covariant (resp., contravariant) cohomological functor.*
3. *For any homomorphism of triangles $(f, g, h) : (X, Y, Z, u, v, w) \to (X', Y', Z', u', v', w')$, if two of $f$, $g$ and $h$ are isomorphisms, then the rest is also an isomorphism.*

*Proof.* 1. According to (TR2), it suffices to show $vu = 0$. By (TR2) and (TR3) we have a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{1_X} & X & \longrightarrow & O & \longrightarrow & TX \\ \| & & \downarrow u & & \downarrow & & \| \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX. \end{array}$$

2. Let $(X, Y, Z, u, v, w)$ be a triangle. Then, since by 1, $vu = 0$, we have $\mathrm{Hom}_{\mathcal{C}}(W, v) \circ \mathrm{Hom}_{\mathcal{C}}(W, u) = 0$. Conversely, let $g \in \mathrm{Hom}_{\mathcal{C}}(W, Y)$ such that $\mathrm{Hom}_{\mathcal{C}}(W, v)(g) = vg = 0$. Then by (TR3) there exists $f \in \mathrm{Hom}_{\mathcal{C}}(W, Y)$ which makes the following diagram commutes

$$\begin{array}{ccccccc} W & \xrightarrow{1_W} & W & \longrightarrow & O & \longrightarrow & TW \\ f\downarrow & & \downarrow g & & \downarrow & & \downarrow Tf \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX. \end{array}$$

Thus $g = \mathrm{Hom}_{\mathcal{C}}(W, u)(f)$ and the sequence $\mathrm{Hom}_{\mathcal{C}}(W, X) \to \mathrm{Hom}_{\mathcal{C}}(W, Y) \to \mathrm{Hom}_{\mathcal{C}}(W, Z)$ is exact. It follows by (TR2) that $\mathrm{Hom}_{\mathcal{C}}(W, -)$ is a cohomological functor.

3. According to (TR2), it is enough to deal with the case that $f$, $g$ are isomorphisms. By 2 we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} \mathrm{Hom}_{\mathcal{C}}(TY', -) & \to & \mathrm{Hom}_{\mathcal{C}}(TX', -) & \to & \mathrm{Hom}_{\mathcal{C}}(Z', -) & \to & \mathrm{Hom}_{\mathcal{C}}(Y', -) & \to & \mathrm{Hom}_{\mathcal{C}}(X', -) \\ \downarrow {\scriptstyle \mathrm{Hom}_{\mathcal{C}}(Tg,-)} & & \downarrow {\scriptstyle \mathrm{Hom}_{\mathcal{C}}(Tf,-)} & & \downarrow {\scriptstyle \mathrm{Hom}_{\mathcal{C}}(h,-)} & & \downarrow {\scriptstyle \mathrm{Hom}_{\mathcal{C}}(g,-)} & & \downarrow {\scriptstyle \mathrm{Hom}_{\mathcal{C}}(f,-)} \\ \mathrm{Hom}_{\mathcal{C}}(TY, -) & \to & \mathrm{Hom}_{\mathcal{C}}(TX, -) & \to & \mathrm{Hom}_{\mathcal{C}}(Z, -) & \to & \mathrm{Hom}_{\mathcal{C}}(Y, -) & \to & \mathrm{Hom}_{\mathcal{C}}(X, -). \end{array}$$

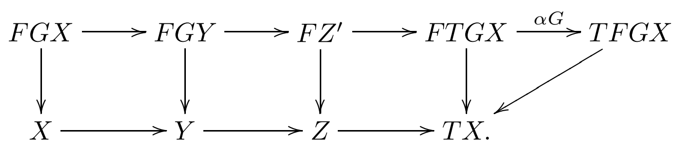
Thus, since by 5 lemma $\mathrm{Hom}_{\mathcal{C}}(h, -)$ is an isomorphism, it follows by Yoneda lemma that $h$ is an isomorphism. □

**Proposition 4.6.** *Let $F : \mathcal{C} \to \mathcal{C}'$ be a $\partial$-functor between triangulated categories. If $G : \mathcal{C}' \to \mathcal{C}$ is a right (resp., left) adjoint of $F$, then $G$ is also a $\partial$-functor.*

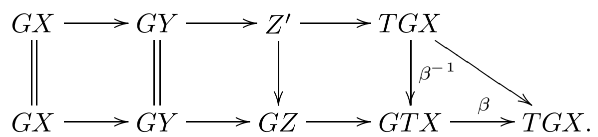
*Proof.* For $X \in \mathcal{C}'$, we have a functorial isomorphisms

$$\begin{aligned} \operatorname{Hom}_{\mathcal{C}}(-, GTX) &\cong \operatorname{Hom}_{\mathcal{C}'}(F-, TX) \\ &\cong \operatorname{Hom}_{\mathcal{C}'}(T^{-1}F-, X) \\ &\cong \operatorname{Hom}_{\mathcal{C}'}(FT^{-1}-, X) \\ &\cong \operatorname{Hom}_{\mathcal{C}}(T^{-1}-, GX) \\ &\cong \operatorname{Hom}_{\mathcal{C}}(-, TGX). \end{aligned}$$

Then we have a functorial isomorphism $\beta : GT_{\mathcal{C}'} \xrightarrow{\sim} T_{\mathcal{C}}G$. For a triangle $(X, Y, Z, u, v, w)$ of $\mathcal{C}'$, let $(GX, GY, Z', Gu, v', w')$ be a triangle of $\mathcal{C}$. Since there is a morphism of triangles




Then we have a commutative diagram




Since $\operatorname{Hom}_{\mathcal{C}}(M, G-) \cong \operatorname{Hom}_{\mathcal{C}'}(FM, -)$, $(GX, GY, GZ, Gu, Gv, Gw)$ induces a long exact sequence. We apply $\operatorname{Hom}_{\mathcal{C}}(M, -)$ to the above diagram, then by 5 lemma, we have an isomorphism from $(GX, GY, Z', Gu, v', w')$ to $(GX, GY, GZ, Gu, Gv, \beta_X Gw)$. □

**Proposition 4.7.** *The following hold.*

1. *If* $\coprod_{i \in I} X_i$ *(resp.,* $\prod_{i \in I} X_i$*) exists in* $\mathcal{C}$ *for* $\{X_i\}_{i \in I}$*, then there is an isomorphism*

$$\begin{gathered} \alpha : \coprod_{i \in I} TX_i \xrightarrow{\sim} T \coprod_{i \in I} X_i \\ \textit{(resp., } \beta : \prod_{i \in I} TX_i \xrightarrow{\sim} T \prod_{i \in I} X_i). \end{gathered}$$

2. *For a collection of triangles* $(X_i, Y_i, Z_i, u_i, v_i, w_i)$ *(*$i \in I$*), if* $\coprod_{i \in I} X_i$*,* $\coprod_{i \in I} Y_i$*,* $\coprod_{i \in I} Y_i$ *(resp.,* $\prod_{i \in I} X_i$*,* $\prod_{i \in I} Y_i$*,* $\prod_{i \in I} Y_i$*) exist in* $\mathcal{C}$*, then*

$$\begin{gathered} (\coprod_{i \in I} X_i, \coprod_{i \in I} Y_i, \coprod_{i \in I} Z_i, \coprod_{i \in I} u_i, \coprod_{i \in I} v_i, \alpha \coprod_{i \in I} w_i) \\ \textit{(resp., } (\prod_{i \in I} X_i, \prod_{i \in I} Y_i, \prod_{i \in I} Z_i, \prod_{i \in I} u_i, \prod_{i \in I} v_i, \beta \prod_{i \in I} w_i)) \end{gathered}$$

*is a triangle.*

*Proof.* 1. We have isomorphisms

$$\begin{aligned} \operatorname{Hom}_{\mathcal{C}}(T \coprod_{i \in I} X_i, -) &\cong \operatorname{Hom}_{\mathcal{C}}(\coprod_{i \in I} X_i, T^{-1}-) \\ &\cong \prod_{i \in I} \operatorname{Hom}_{\mathcal{C}}(X_i, T^{-1}-) \\ &\cong \prod_{i \in I} \operatorname{Hom}_{\mathcal{C}}(TX_i, -) \\ &\cong \operatorname{Hom}_{\mathcal{C}}(\coprod_{i \in I} TX_i, -). \end{aligned}$$

2. Let $(\coprod_{i\in I}X_i, \coprod_{i\in I}Y_i, Z', \coprod_{i\in I}u_i, v, w)$ be a triangle. Then we have a commutative diagram

$$\begin{array}{ccccccc}
\coprod_{i\in I}X_i & \xrightarrow{\coprod_{i\in I}u_i} & \coprod_{i\in I}Y_i & \xrightarrow{\coprod_{i\in I}v_i} & \coprod_{i\in I}Z_i & \xrightarrow{\alpha\coprod_{i\in I}v_i} & T\coprod_{i\in I}X_i \\
\| & & \| & & \downarrow & & \| \\
\coprod_{i\in I}X_i & \xrightarrow{\coprod_{i\in I}u_i} & \coprod_{i\in I}Y_i & \xrightarrow{v} & Z' & \xrightarrow{w} & T\coprod_{i\in I}X_i.
\end{array}$$

Applying $\mathrm{Hom}_{\mathcal{C}}(M, -)$ to the above, by 5 lemma, we complete the proof. $\square$

**Proposition 4.8.** *The following hold.*

1. *A triangle* $(X, Y, Z, u, v, 0)$ *is isomorphic to* $(X, Z\oplus X, Z, \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}1 & 0\end{smallmatrix}\right], 0)$.
2. *For a morphism of triangles*

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\
\| & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \| \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & TX
\end{array}$$

*there exists* $g' : Z \to Z'$ *such that*

$$Y \xrightarrow{\left[\begin{smallmatrix}v\\f\end{smallmatrix}\right]} Z\oplus Y' \xrightarrow{\left[\begin{smallmatrix}-g' & v'\end{smallmatrix}\right]} Z' \xrightarrow{(Tu)w'} TY$$

*is a triangle.*

*Proof.* 1. Since $\mathrm{Hom}_{\mathcal{C}}(Z, Z) \xrightarrow{0} \mathrm{Hom}_{\mathcal{C}}(Z, TX)$, by Proposition 4.5, there is $s : Z \to Y$ such that $vs = 1_Z$. Then we have a commutative diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{\mu} & Z\oplus X & \xrightarrow{\pi} & Z & \xrightarrow{0} & TX \\
\| & & {\scriptstyle \alpha}\downarrow & & \| & & \| \\
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{0} & TX
\end{array}$$

where $\mu = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, $\pi = \left[\begin{smallmatrix}1 & 0\end{smallmatrix}\right]$, $\alpha = \left[\begin{smallmatrix}s & u\end{smallmatrix}\right]$.

2. Since $T^{-1}Z' \xrightarrow{-T^{-1}w'} X \xrightarrow{u'} Y' \xrightarrow{v'} Z'$ is triangle, we have a commutative diagram

$$\begin{array}{ccccccc}
T^{-1}Z' & \xrightarrow{-T^{-1}w'} & X & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' \\
\| & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle x} & & \| \\
T^{-1}Z' & \xrightarrow{-uT^{-1}w'} & Y & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & Z' \\
 & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle -w'} \\
 & & Z & = & Z & \xrightarrow{w} & TX \\
 & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle 0} & & \\
 & & TX & \xrightarrow{Tu'} & TY' & &
\end{array}$$

By 1, $Y' \xrightarrow{x} M \xrightarrow{y} Z \xrightarrow{0} TY'$ is isomorphic to $Y' \xrightarrow{\mu} Z \oplus Y' \xrightarrow{\pi} Z \xrightarrow{0} TY'$. Then we have a commutative diagram

$$\begin{array}{ccccccc}
T^{-1}Z' & \xrightarrow{-T^{-1}w'} & X & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' \\
\| & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \mu} & & \| \\
T^{-1}Z' & \xrightarrow{-uT^{-1}w'} & Y & \xrightarrow{\alpha} & Z\oplus Y' & \xrightarrow{\beta} & Z' \\
 & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle -w'} \\
 & & Z & = & Z & \xrightarrow{w} & TX \\
 & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle 0} & & \\
 & & TX & \xrightarrow{Tu'} & TY' & &
\end{array}$$

where $\mu = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$, $\pi = \left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right]$, $\alpha = \left[\begin{smallmatrix} v \\ f' \end{smallmatrix}\right]$, $\beta = \left[\begin{smallmatrix} -g'' & v' \end{smallmatrix}\right]$, and $v'f' = g''v$, $f'u = u'$, $w = w'g'$. Since $(f' - f)u = 0$, there is $h : Z \to Y'$ such that $f' = f + hv$. Hence we have a commutative diagram

$$\begin{array}{ccccccc}
Y & \xrightarrow{\alpha'} & Z\oplus Y' & \xrightarrow{\beta'} & Z & \xrightarrow{(Tu)w'} & TY \\
\| & & {\scriptstyle \phi}\downarrow & & \| & & \| \\
Y & \xrightarrow{\alpha} & Z\oplus Y' & \xrightarrow{\beta} & Z & \xrightarrow{(Tu)w'} & TY
\end{array}$$

where $\alpha' = \left[\begin{smallmatrix} v \\ f \end{smallmatrix}\right]$, $\beta' = \left[\begin{smallmatrix} -g' & v' \end{smallmatrix}\right]$, $\phi = \left[\begin{smallmatrix} 1 & 0 \\ h & 1 \end{smallmatrix}\right]$. $\square$

**Proposition 4.9** (9 Lemma)**.** *Any commutative diagram in $\mathcal{C}$*

$$\begin{array}{ccc}
X' & \xrightarrow{u'} & Y' \\
{\scriptstyle x'}\downarrow & & \downarrow{\scriptstyle y'} \\
X & \xrightarrow{u} & Y
\end{array}$$

*can be embedded in a diagram*

$$\begin{array}{ccccccc}
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & TX' \\
\downarrow{\scriptstyle x'} & & \downarrow{\scriptstyle y'} & & \downarrow{\scriptstyle z'} & & \downarrow{\scriptstyle Tx'} \\
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\
\downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle z} & & \downarrow{\scriptstyle Tx} \\
X'' & \xrightarrow{u''} & Y'' & \xrightarrow{v''} & Z'' & \xrightarrow{w''} & TX'' \\
\downarrow{\scriptstyle x''} & & \downarrow{\scriptstyle y''} & & \downarrow{\scriptstyle z''} & - & \downarrow{\scriptstyle -Tx''} \\
TX' & \xrightarrow{Tu'} & TY' & \xrightarrow{Tv'} & TZ' & \xrightarrow{-Tw'} & T^2X'
\end{array} .$$

*which is commutative without the right and bottom corner, - anti-commutative, where all rows and columns are triangles.*

*Proof.* According to (TR4), we have three commutative diagrams

$$\begin{array}{ccccccc}
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & TX' \\
\| & & \downarrow{\scriptstyle y'} & & \downarrow{\scriptstyle \gamma} & & \| \\
X' & \xrightarrow{y'u'} & Y & \xrightarrow{\alpha} & A & \xrightarrow{\beta} & TX' \\
 & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle Tu'} \\
 & & Y'' & = & Y'' & \xrightarrow{y''} & TY' \\
 & & \downarrow{\scriptstyle y''} & & \downarrow{\scriptstyle (Tv')y''} & & \\
 & & TY' & \xrightarrow{Tv'} & TZ' & &
\end{array}$$

$$\begin{array}{ccccccc}
X' & \xrightarrow{x'} & X & \xrightarrow{x} & X'' & \xrightarrow{x''} & TX' \\
\| & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \varepsilon} & & \| \\
X' & \xrightarrow{ux'} & Y & \xrightarrow{\alpha} & A & \xrightarrow{\beta} & TX' \\
 & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle Tx'} \\
 & & Z & = & Z & \xrightarrow{w} & TX \\
 & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle (Tx)w} & & \\
 & & TX & \xrightarrow{Tx} & TX'' & &
\end{array}$$

$$\begin{array}{ccccccc}
X'' & \xrightarrow{\varepsilon} & A & \xrightarrow{\eta} & Z & \xrightarrow{(Tx)w} & TX'' \\
\| & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle z} & & \| \\
X'' & \xrightarrow{u''} & Y'' & \xrightarrow{v''} & Z'' & \xrightarrow{w''} & TX \\
 & & \downarrow{\scriptstyle (Tv')y''} & & \downarrow{\scriptstyle z''} & & \downarrow{\scriptstyle T\varepsilon} \\
 & & TZ' & = & TZ' & \xrightarrow{-T\gamma} & TA \\
 & & \downarrow{\scriptstyle -T\gamma} & & \downarrow{\scriptstyle -Tz''} & & \\
 & & TA & \xrightarrow{T\eta} & TZ & &
\end{array}$$

In particular, we have $u'' = \delta\varepsilon$, $v = \eta\alpha$, $y = \delta\alpha$, $z' = \eta\gamma$, $z\eta = v''\delta$ and $(Tx'')w'' = (T\beta)(T\varepsilon)w'' = -(T\beta)(T\gamma)z'' = -(Tw')z'$. Then it is easy to get the diagram. □

## 5. Frobenius Categories

**Definition 5.1** (Exact Category)**.** Let $\mathcal{C}$ be an additive category which is embedded as a full subcategory of an abelian category $\mathcal{A}$, and suppose that $\mathcal{C}$ is closed under extensions in $\mathcal{A}$. Let $\mathcal{S}$ be a collection of exact sequences in $\mathcal{A}$

$$O \to X \xrightarrow{u} Y \xrightarrow{v} Z \to O.$$

$u$ is called an *admissible monomorphism*, and $v$ is called an *admissible epimorphism*.

A pair $(\mathcal{C}, \mathcal{S})$ is called an *exact category* in the sense of Quillen provided that

(EX1) Any split sequence of which all terms are in $\mathcal{C}$ is in $\mathcal{S}$.

(EX2) The composition of admissible monomorphisms (resp., epimorphisms) is also an admissible monomorphism (resp., epimorphism).

(EX3) Given the following commutative diagram in $\mathcal{A}$

$$\begin{array}{ccccccccc} O & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & O \\ & & \downarrow & & \downarrow & & \| & & \\ O & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & Z & \longrightarrow & O \end{array}$$

where all rows are exact, if the top row is in $\mathcal{S}$ and $X' \in \mathcal{C}$, then the bottom row is in $\mathcal{S}$.

(EX4) Given the following commutative diagram in $\mathcal{A}$

$$\begin{array}{ccccccccc} O & \longrightarrow & X & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & O \\ & & \| & & \downarrow & & \downarrow & & \\ O & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & O \end{array}$$

where all rows are exact, if the bottom row is in $\mathcal{S}$ and $Z' \in \mathcal{C}$, then the top row is in $\mathcal{S}$.

An object $X$ in $\mathcal{C}$ is called *$\mathcal{S}$-projective* (resp., *$\mathcal{S}$-injective*) if for any admissible epimorphisms (resp., monomorphisms) $v : Y \to Z$, $\mathrm{Hom}_{\mathcal{C}}(X, v)$ (resp., $\mathrm{Hom}_{\mathcal{C}}(v, X)$) is surjective.

**Definition 5.2** (Frobenius Category)**.** An exact category $(\mathcal{C}, \mathcal{S})$ is called a *Frobenius category* if $(\mathcal{C}, \mathcal{S})$ has enough $\mathcal{S}$-projectives and enough $\mathcal{S}$-injectives and if $\mathcal{S}$-projectives coincide with $\mathcal{S}$-injectives.

Let $\mathcal{Q}$ be the full subcategory of $\mathcal{C}$ consisting of $\mathcal{S}$-projective objects. A stable category $\underline{\mathcal{C}}$ is the category $\mathcal{C}_{\mathcal{Q}}$.

**Proposition 5.3.** *In a Frobenius category $(\mathcal{C}, \mathcal{S})$, we consider the following commutative diagram*

$$\begin{array}{ccccccccc} O & \longrightarrow & X & \longrightarrow & I & \longrightarrow & X_1 & \longrightarrow & O \\ & & {\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle f'} & & \\ O & \longrightarrow & X' & \longrightarrow & I' & \longrightarrow & X'_1 & \longrightarrow & O \end{array}$$

*where $I$, $I'$ are $\mathcal{S}$-injective, with all rows in $\mathcal{S}$. Then the image $\underline{f'}$ is uniquely determined by $f$ in $\underline{\mathcal{C}}$.*

**Remark 5.4.** For all $X \in \mathcal{C}$ we choose the elements $O \to X \xrightarrow{\mu_X} I(X) \xrightarrow{\pi_X} TX \to O$ in $\mathcal{S}$, with $I(X)$ being $\mathcal{S}$-injective. According to Proposition 5.3, an object $TX$ is uniquely determined up to isomorphism in $\underline{\mathcal{C}}$ independently of choice of the above sequence, but $\underline{f'}$ is depend on their choice. Then we can understand the induced functor $T : \underline{\mathcal{C}} \to \underline{\mathcal{C}}$ only if we know $O \to X \xrightarrow{\mu_X} I(X) \xrightarrow{\pi_X} TX \to O$ in $\mathcal{S}$ for all $X \in \mathcal{C}$.

**Proposition 5.5.** *$T$ is an auto-equivalence of $\underline{\mathcal{C}}$.*

**Definition 5.6** (Triangle)**.** In a Frobenius category $(\mathcal{C}, \mathcal{S})$, let $u : X \to Y$ be an morphism in $\mathcal{C}$. By taking $\mathrm{M}(u) = \mathrm{PushOut}(u, \mu_X)$, we have the following commutative diagram in $\mathcal{C}$

$$\begin{array}{ccccccccc} O & \longrightarrow & X & \xrightarrow{\mu_X} & I(X) & \xrightarrow{\pi_X} & TX & \longrightarrow & O \\ & & {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle x} & & \| & & \\ O & \longrightarrow & Y & \xrightarrow{v} & \mathrm{M}(u) & \xrightarrow{w} & TX & \longrightarrow & O \end{array}$$

with all rows in $\mathcal{S}$. Then in $\underline{\mathcal{C}}$ the sequence

$$X \xrightarrow{u} Y \xrightarrow{v} \mathrm{M}(u) \xrightarrow{w} TX$$

is called a *standard triangle*. Let $\mathcal{T}$ be a collection of sextuples which are isomorphic to standard triangles in $\underline{\mathcal{C}}$.

**Lemma 5.7.** *In a Frobenius category $(\mathcal{C}, \mathcal{S})$, let $w : \mathrm{M}(u) \to TX$ be the morphism in Definition 5.6. Consider the following commutative diagram in $\mathcal{C}$*

$$\begin{array}{ccccccccc}
O & \longrightarrow & X & \xrightarrow{-u'} & Y' & \xrightarrow{v'} & \mathrm{M}(u) & \longrightarrow & O \\
 & & \| & & \downarrow{\scriptstyle x'} & & \downarrow{\scriptstyle w} & & \\
O & \longrightarrow & X & \xrightarrow{\mu} & I(X) & \xrightarrow{\pi} & TX & \longrightarrow & O
\end{array}$$

*with all rows in $\mathcal{S}$, then the sextuple $X \xrightarrow{u'} Y' \xrightarrow{v'} \mathrm{M}(u) \xrightarrow{w} TX$ is isomorphic to the above triangle $X \xrightarrow{u} Y \xrightarrow{v} \mathrm{M}(u) \xrightarrow{w} TX$ in $\underline{\mathcal{C}}$.*

*Proof.* By Proposition 2.21, we have the following commutative diagram

$$\begin{array}{ccccccccc}
O & \longrightarrow & X & \xrightarrow{\beta} & I(X) \oplus Y & \xrightarrow{\gamma} & \mathrm{M}(u) & \longrightarrow & O \\
 & & \| & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle w} & & \\
O & \longrightarrow & X & \xrightarrow{\mu} & I(X) & \xrightarrow{-\pi} & TX & \longrightarrow & O
\end{array}$$

where $\beta = \left[\begin{smallmatrix} \mu \\ u \end{smallmatrix}\right]$, $\gamma = \left[\begin{smallmatrix} -x & v \end{smallmatrix}\right]$, $\delta = \left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right]$, with all rows in $\mathcal{S}$. Since the right and bottom rectangle in the previous diagram is pull back, there exists $\eta : Y' \to I(X) \oplus Y$ such that we have the following commutative diagram in $\underline{\mathcal{C}}$

$$\begin{array}{ccccccc}
X & \xrightarrow{u'} & Y' & \xrightarrow{v'} & \mathrm{M}(u) & \xrightarrow{w} & TX \\
\| & & \downarrow{\scriptstyle \eta} & & \| & & \| \\
X & \xrightarrow{\beta} & I(X) \oplus Y & \xrightarrow{\gamma} & \mathrm{M}(u) & \xrightarrow{w} & TX \\
\| & & \downarrow{\scriptstyle \varepsilon} & & \| & & \| \\
X & \xrightarrow{u} & Y & \xrightarrow{v} & \mathrm{M}(u) & \xrightarrow{w} & TX
\end{array}$$

where $\varepsilon = \left[\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right]$, all vertical arrows are isomorphisms in $\underline{\mathcal{C}}$. $\square$

**Proposition 5.8.** *In a Frobenius category $(\mathcal{C}, \mathcal{S})$, the image of any element $O \to X \xrightarrow{u} Y \xrightarrow{v} Z \to O$ of $\mathcal{S}$ can be embedded in a triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ in $\underline{\mathcal{C}}$.*

*Proof.* Since $I(X)$ is $\mathcal{S}$-injective and $O \to X \xrightarrow{-u} Y \xrightarrow{v} Z \to O \in \mathcal{S}$, we have a commutative diagram

$$\begin{array}{ccccccccc}
O & \longrightarrow & X & \xrightarrow{-u} & Y & \xrightarrow{v} & Z & \longrightarrow & O \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle w} & & \\
O & \longrightarrow & X & \xrightarrow{\mu} & I(X) & \xrightarrow{\pi} & TX & \longrightarrow & O.
\end{array}$$

By Lemma 5.7, we get the statement. $\square$

**Theorem 5.9.** *Let $(\mathcal{C}, \mathcal{S})$ be a Frobenius category. Then $(\underline{\mathcal{C}}, \mathcal{T})$ is a triangulated category.*

*Proof.* we show that $(\mathcal{C}, \mathcal{S})$ satisfies the axioms of a triangulated category.

(TR1) It is trivial.

(TR2) Let $(X, Y, Z, \underline{u}, \underline{v}, \underline{w})$ be a standard triangle. Then we have a commutative diagram

$$\begin{array}{ccccccccc}
O & \longrightarrow & X & \xrightarrow{\mu_X} & I(X) & \xrightarrow{\pi_X} & TX & \longrightarrow & O \\
 & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle x} & & \| & & \\
O & \longrightarrow & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX & \longrightarrow & O \\
 & & \downarrow{\scriptstyle \mu_Y} & & \downarrow{\scriptstyle \alpha} & & \| & & \\
O & \longrightarrow & I(Y) & \xrightarrow{\beta} & I(Y)\oplus TX & \xrightarrow{\gamma} & TX & \longrightarrow & O \\
 & & \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle \delta} & & & & \\
 & & TY & = & TY & & & &
\end{array}$$

where $\alpha = \left[\begin{smallmatrix} s \\ w \end{smallmatrix}\right]$, $\beta = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, $\gamma = \left[\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right]$, $\delta = \left[\begin{smallmatrix} \pi_Y & u' \end{smallmatrix}\right]$, $\pi_Y s + u'w = 0$, and with all rows in $\mathcal{S}$. Since $sx\mu_X = \left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right] \alpha x \mu_X = \left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right] \beta \mu_Y u = \mu_Y u$, we have a commutative diagram

$$\begin{array}{ccccccccc}
O & \longrightarrow & X & \xrightarrow{\mu_X} & I(X) & \xrightarrow{\pi_X} & TX & \longrightarrow & O \\
 & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle sx} & & \downarrow{\scriptstyle -u'} & & \\
O & \longrightarrow & Y & \xrightarrow{\mu_Y} & I(Y) & \xrightarrow{\pi_Y} & TY & \longrightarrow & O.
\end{array}$$

Then we have $\underline{u'} = -T\underline{u}$, and we have a commutative diagram in $\underline{\mathcal{C}}$

$$\begin{array}{ccccccc}
Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX & \xrightarrow{u'} & TY \\
\| & & \| & & \downarrow{\scriptstyle \epsilon} & & \| \\
Y & \xrightarrow{v} & Z & \xrightarrow{\alpha} & I(Y)\oplus TX & \xrightarrow{\delta} & TY
\end{array}$$

where $\varepsilon = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ is an isomorphism in $\underline{\mathcal{C}}$. Hence a sextuple $(Y, Z, TX, \underline{v}, \underline{w}, -T\underline{u})$ is a triangle . The reverse implication is similar by Lemma 5.7.

(TR3) Let $(X_i, Y_i, Z_i, \underline{u_i}, \underline{v_i}, \underline{w_i})$, be standard triangles $(i = 1, 2)$, and

$$\begin{array}{ccccccccc}
O & \longrightarrow & X_i & \xrightarrow{\mu_{X_i}} & I(X_i) & \xrightarrow{\pi_{X_i}} & TX_i & \longrightarrow & O \\
 & & {\scriptstyle u_i}\downarrow & & \downarrow{\scriptstyle x_i} & & \| & & \\
O & \longrightarrow & Y_i & \xrightarrow{v_i} & Z_i & \xrightarrow{w_i} & TX_i & \longrightarrow & O
\end{array}$$

commutative diagrams with all rows in $\mathcal{S}$. Let $f : X_1 \to X_2$, $g : Y_1 \to Y_2$ be morphisms satisfying $\underline{u_2}\underline{f} = \underline{g}\underline{u_1}$ in $\underline{\mathcal{C}}$. Since $I(X_1)$ is $\mathcal{S}$-injective, there exists $t : I(X_1) \to Y_2$ such that $gu_1 - u_2 f = t\mu_{X_1}$. Since $I(X_2)$ is $\mathcal{S}$-injective, we have a commutative diagram

$$\begin{array}{ccc}
X_1 & \xrightarrow{\mu_{X_1}} & I(X_1) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle r} \\
X_2 & \xrightarrow{\mu_{X_2}} & I(X_2).
\end{array}$$

Then we have equations

$$\begin{aligned} x_2 r\mu_{X_1} &= x_2\mu_{X_2} f \\ &= v_2 u_2 f \end{aligned}$$

$$\begin{aligned} v_2 g u_1 &= v_2 u_2 f + v_2 t\mu_{X_1} \\ &= (x_2 r + v_2 t)\mu_{X_1} \end{aligned}$$

Since

$$\begin{array}{ccc} X_1 & \xrightarrow{\mu_{X_1}} & I(X_1) \\ {\scriptstyle u_1}\downarrow & & \downarrow{\scriptstyle x_1} \\ Y_1 & \xrightarrow{v_1} & Z_1 \end{array}$$

is push out, there exists $h : \mathrm{M}(u_1) \to \mathrm{M}(u_2)$ such that $v_2 g = hv_1$ and $x_2 r + v_2 t = hx_1$. Then there is $f' : TX_1 \to TX_2$ such that $f'w_1 = w_2 h$. Since

$$f'\pi_{X_1} = f'w_1 x_1 = w_2 h x_1 = w_2(x_2 r + v_2 t) = w_2 x_2 r = \pi_{X_2} r$$

we have $\underline{f'} = T\underline{f}$, and hence a morphism $(\underline{f}, \underline{g}, \underline{h})$ from $(X_1, Y_1, Z_1, \underline{u_1}, \underline{v_1}, \underline{w_1})$ to $(X_2, Y_2, Z_2, \underline{u_2}, \underline{v_2}, \underline{w_2})$.

(TR4) Let $(X, Y, Z', \underline{u}, \underline{i}, \underline{i'})$, $(X, Z, Y', \underline{vu}, \underline{k}, \underline{k'})$ and $(Y, Z, X', \underline{v}, \underline{j}, \underline{j'})$ be triangles in $\underline{\mathcal{C}}$. We have a commutative diagram in $\mathcal{C}$

$$\begin{array}{ccccccccc} O & \longrightarrow & X & \xrightarrow{\mu_X} & I(X) & \xrightarrow{\pi_X} & TX & \longrightarrow & O \\ & & {\scriptstyle u}\downarrow & EX & \downarrow{\scriptstyle x} & & \| & & \\ O & \longrightarrow & Y & \xrightarrow{i} & Z' & \xrightarrow{i'} & TX & \longrightarrow & O \\ & & {\scriptstyle v}\downarrow & EX & \downarrow{\scriptstyle f} & & \| & & \\ O & \longrightarrow & Z & \xrightarrow{k} & Y' & \xrightarrow{k'} & TX & \longrightarrow & O \end{array}$$

where all rows are in $\mathcal{S}$. Since $i$ is an admissible monomorphism, $\mu'_Y = \mu_{Z'} i$ is also an admissible monomorphism and there is an admissible epimorphism $\pi'_Y$ such that $\pi_{Z'} = (T'i)\pi'_Y$. Then we have a commutative diagram in $\mathcal{C}$

$$\begin{array}{ccccccccccc} X & \xrightarrow{\mu_X} & I(X) & \xrightarrow{\pi_X} & TX & & & & & & \\ {\scriptstyle u}\downarrow & EX & \downarrow{\scriptstyle x} & & & \searrow^{T'u} & & & & & \\ Y & \xrightarrow{i} & Z' & \xrightarrow{\mu_{Z'}} & I(Z') & \xrightarrow{\pi'_Y} & T'Y & \xrightarrow{T'i} & TZ' \\ {\scriptstyle v}\downarrow & EX & \downarrow{\scriptstyle f} & EX & \downarrow{\scriptstyle z'} & & \| & & \| \\ Z & \xrightarrow{k} & Y' & \xrightarrow{g_1} & X'' & \xrightarrow{j'_1} & T'Y & \xrightarrow{T'i} & TZ' \end{array}$$

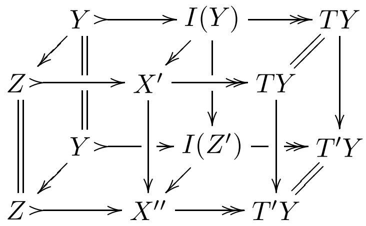
Therefore, we have a triangle in $\underline{\mathcal{C}}$

$$Z' \xrightarrow{\underline{f}} Y' \xrightarrow{\underline{g_1}} X' \xrightarrow{\underline{(T'i)j'_1}} TZ'.$$

Since $j'_1 g_1 f x = (T'u)\pi_X = (T'u)k'fx$ and $j'_1 g_1 k = 0 = (T'u)k'k$ in $\mathcal{C}$, and

$$\begin{array}{ccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z \\ {\scriptstyle \mu_X}\downarrow & & & & \downarrow{\scriptstyle k} \\ I(X) & \xrightarrow{x} & Z' & \xrightarrow{f} & Y' \end{array}$$

is push out, we have $j_1'g_1 = (T'u)k'$. Since $I(Y)$ and $I(Z')$ are $\mathcal{S}$-injective, there exist $\alpha : TY \to T'Y$ and $\beta : X' \to X''$ such that



is commutative in $\mathcal{C}$, where $\underline{\alpha}$ is an isomorphism in $\underline{\mathcal{C}}$. Then by Proposition 2.22, $\underline{\beta}$ is an isomorphism in $\underline{\mathcal{C}}$, and $T\underline{i} = (\underline{T'i})\underline{\alpha}$, $T\underline{u} = \underline{\alpha}^{-1}\underline{T'u}$. Let $\underline{g} = \underline{\beta}^{-1}\underline{g_1}$, $\underline{j} = \underline{\beta}^{-1}\underline{g_1 k}$, $\underline{j'} = \underline{\alpha}^{-1}\underline{j_1'\beta}$, then we have the octahedral diagram of Definition 4.1. □

**Example 5.10.** Let $A$ be a self-injective algebra over a field $k$, and $O \to \Omega \to A\otimes_k A \xrightarrow{\mu} A \to O$ an exact sequence, where $\mu$ is the multiplication map. Then $\mathsf{mod}\,A$ is a Frobenius category, its stable category $\underline{\mathsf{mod}}A$ is a triangulated category with a translation functor $\mathrm{Hom}_A(\Omega, -)$.

## 6. Homotopy Categories

Throughout this section, $\mathcal{A}$ is an abelian category and $\mathcal{B}$ is an additive subcategory of $\mathcal{A}$ which is closed under isomorphisms.

**Definition 6.1** (Complex)**.** Let $\mathcal{B}$ be an additive category. A *complex* (*cochain complex*) is a collection $X^\cdot = (X^n, d_X^n : X^n \to X_X^{n+1})_{n\in\mathbb{Z}}$ of objects and morphisms of $\mathcal{B}$ such that $d_X^{n+1}d_X^n = 0$. A complex $X^\cdot = (X^n, d_X^n : X^n \to X_X^{n+1})_{n\in\mathbb{Z}}$ is called *bounded below* (resp., *bounded above*, *bounded*) if $X^n = O$ for sufficiently small (resp., large, large and small) $n$.

A complex $X^\cdot = (X^n, d_X^n)$ is called a *stalk complex* if there exists an integer $n_0$ such that $X^i = O$ if $i \neq n_0$. We identify objects of $\mathcal{B}$ with a stalk complexes of degree 0.

A *morphism* $f$ of *complexes* $X^\cdot$ to $Y^\cdot$ is a collection of morphisms $f^n : X^n \to Y^n$ which commute with the maps of complexes

$$f^{n+1}d_X^n = d_Y^n f^n.$$

We denote by $\mathsf{C}(\mathcal{B})$ (resp., $\mathsf{C}^+(\mathcal{B})$, $\mathsf{C}^-(\mathcal{B})$, $\mathsf{C}^{\mathrm{b}}(\mathcal{B})$) the category of complexes (resp., bounded below complexes, bounded above complexes, bounded complexes) of $\mathcal{B}$. An auto-equivalence $T : \mathsf{C}(\mathcal{B}) \to \mathsf{C}(\mathcal{B})$ is called translation if $(TX^\cdot)^n = X^{n+1}$ and $(Td_X)^n = -d_X^{n+1}$ for any complex $X^\cdot = (X^n, d_X^n)$.

**Proposition 6.2.** *The following hold.*

1. $\mathsf{C}^*(\mathcal{B})$ *is an additive category, where* $* =$ *nothing,* $+, -, b$*. Moreover, if* $\mathcal{B}$ *has products (resp., coproducts), then* $\mathsf{C}(\mathcal{B})$ *has also products (resp., coproducts).*
2. $\mathsf{C}^*(\mathcal{A})$ *is an abelian category, where* $* =$ *nothing,* $+, -, b$*. Moreover, if* $\mathcal{A}$ *satisfies the condition Ab3* (resp., Ab3), then* $\mathsf{C}(\mathcal{A})$ *also satisfies the condition Ab3* (resp., Ab3).*

**Definition 6.3.** For $u \in \mathrm{Hom}_{\mathsf{C}(\mathcal{B})}(X^\cdot, Y^\cdot)$, the *mapping cone* of $u$ is a complex $\mathrm{M}^\cdot(u)$ with

$$\mathrm{M}^n(u) = X^{n+1} \oplus Y^n,$$
$$d^n_{\mathrm{M}^\cdot(u)} = \begin{bmatrix} -d_X^{n+1} & 0 \\ u^{n+1} & d_X^n \end{bmatrix} : X^{n+1} \oplus Y^n \to X^{n+2} \oplus Y^{n+1}.$$

Moreover, for $1_X \in \mathrm{Hom}_{\mathsf{C}(\mathcal{B})}(X^\cdot, X^\cdot)$, let $I^\cdot(X^\cdot) = \mathrm{M}^\cdot(1_X)$.

**Definition 6.4.** Let $\mathcal{S}_{\mathsf{C}^*(\mathcal{B})}$ be the collection of exact sequences $O \to X^\cdot \xrightarrow{f} Y^\cdot \xrightarrow{g} Z^\cdot \to O$ of complexes of $\mathsf{C}^*(\mathcal{B})$ such that

$$O \to X^n \xrightarrow{f^n} Y^n \xrightarrow{g^n} Z^n \to O$$

are split exact for all $n \in \mathbb{Z}$, where $* =$ nothing, $+, -, b$. In this case, we call $f$ (resp., $g$) a *term-split monomorphism* (resp., a *term-split epimorphism*).

**Proposition 6.5.** *For $u \in \mathrm{Hom}_{\mathsf{C}(\mathcal{B})}(X^\cdot, Y^\cdot)$, we have an exact sequence in $\mathsf{C}(\mathcal{A})$*

$$O \to Y^\cdot \xrightarrow{\mu_u} \mathrm{M}^\cdot(u) \xrightarrow{\pi_u} TX^\cdot \to O$$

*where $\mu_u = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$, $\pi_u = [\,1\ 0\,]$. Moreover, the above sequence belongs to $\mathcal{S}_{\mathsf{C}(\mathcal{B})}$.*

**Lemma 6.6.** *For $X^\cdot \in \mathsf{C}(\mathcal{B})$, we have*

$$I^\cdot(X^\cdot) \cong (X^{n+1} \oplus X^n, \left[\begin{smallmatrix} 0 & 0 \\ 1_{X^{n+1}} & 0 \end{smallmatrix}\right] : X^{n+1} \oplus X^n \to X^{n+2} \oplus X^{n+1}).$$

*Proof.*

$$\begin{array}{ccc} X^{n+1} \oplus X^n & \xrightarrow{d^n_{\mathrm{M}^\cdot(1_X)}} & X^{n+2} \oplus X^{n+1} \\ {\scriptstyle \alpha^n}\downarrow & & \downarrow{\scriptstyle \alpha^{n+1}} \\ X^{n+1} \oplus X^n & \xrightarrow{\delta^n} & X^{n+2} \oplus X^{n+1} \end{array}$$

where $\alpha^n = \begin{bmatrix} 1_{X^{n+1}} & d_X^n \\ 0 & 1_{X^n} \end{bmatrix}$ and $\delta^n = \left[\begin{smallmatrix} 0 & 0 \\ 1_{X^{n+1}} & 0 \end{smallmatrix}\right]$. □

**Lemma 6.7.** *The category $(\mathsf{C}(\mathcal{B}), \mathcal{S}_{\mathsf{C}(\mathcal{B})})$ is an exact category.*

**Proposition 6.8.** *The category $(\mathsf{C}^*(\mathcal{B}), \mathcal{S}_{\mathsf{C}^*(\mathcal{B})})$ is a Frobenius category, where $* =$ nothing, $+, -, b$.*

*Proof.* Let $X^\cdot \in \mathsf{C}(\mathcal{B})$, then by Lemma 6.6 we have

$$I^\cdot(X^\cdot) \cong \bigoplus_{n \in \mathbb{Z}} I^\cdot(X^n)[-n].$$

where $X^n$ is a stalk complex of degree 0 (Note that the above biproduct exists by Exercise 6.18). It is easy to see that $I^\cdot(X^n)[-n]$ is $\mathcal{S}_{\mathsf{C}(\mathcal{B})}$-projective and $\mathcal{S}_{\mathsf{C}(\mathcal{B})}$-injective, and then $I^\cdot(X^\cdot)$ is $\mathcal{S}_{\mathsf{C}(\mathcal{B})}$-projective and $\mathcal{S}_{\mathsf{C}(\mathcal{B})}$-injective. For any $\mathcal{S}_{\mathsf{C}(\mathcal{B})}$-injective complex $Y^\cdot$, by Proposition 6.5, $Y^\cdot$ is a direct summand of $I^\cdot(Y^\cdot)$, and hence $Y^\cdot$ is $\mathcal{S}_{\mathsf{C}(\mathcal{B})}$-projective. Similarly, any $\mathcal{S}_{\mathsf{C}(\mathcal{B})}$-projective complex is $\mathcal{S}_{\mathsf{C}(\mathcal{B})}$-injective. According to Proposition 6.5, it is easy to see that $\mathsf{C}(\mathcal{B})$ has enough $\mathcal{S}_{\mathsf{C}(\mathcal{B})}$-injectives and enough $\mathcal{S}_{\mathsf{C}(\mathcal{B})}$-projectives. □

**Definition 6.9** (Homotopy Category)**.** A *homotopy category* $\mathsf{K}^*(\mathcal{B})$ of $\mathcal{B}$ is the stable category of $(\mathsf{C}^*(\mathcal{B}), \mathcal{S}_{\mathsf{C}^*(\mathcal{B})})$ by the full subcategory $\mathcal{I}_{\mathsf{C}^*(\mathcal{B})}$ of $\mathcal{S}_{\mathsf{C}^*(\mathcal{B})}$-injective objects, where $* =$ nothing, $+, -, b$.

**Remark 6.10.** For $u \in \mathrm{Hom}_{\mathsf{C}(\mathcal{B})}(X^{\bullet}, Y^{\bullet})$, we have a commutative diagram

$$\begin{array}{ccccccccc} O & \longrightarrow & X^{\bullet} & \xrightarrow{\mu_X} & I^{\bullet}(X^{\bullet}) & \xrightarrow{\pi_X} & TX^{\bullet} & \longrightarrow & O \\ & & {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle x} & & \| & & \\ O & \longrightarrow & Y^{\bullet} & \xrightarrow{\mu_u} & \mathrm{M}^{\bullet}(u) & \xrightarrow{\pi_u} & TX^{\bullet} & \longrightarrow & O \end{array}$$

where $x = \left[\begin{smallmatrix} 1 & 0 \\ 0 & u \end{smallmatrix}\right]$, with all rows in $\mathcal{S}_{\mathsf{C}(\mathcal{B})}$. By the proof of Proposition 6.8, the definition of $\mathrm{M}^{\bullet}(u)$ coincides with the one of Definition 5.6.

**Proposition 6.11.** *A category $\mathsf{K}^*(\mathcal{B})$ is a triangulated category, where $* =$ nothing, $+, -, b$.*

**Proposition 6.12.** *If an exact sequence $O \to X^{\bullet} \xrightarrow{u} Y^{\bullet} \xrightarrow{v} Z^{\bullet} \to O$ in $\mathsf{C}^*(\mathcal{B})$ belongs to $\mathcal{S}_{\mathsf{C}^*(\mathcal{B})}$, then it can be embedded in a triangle $X^{\bullet} \xrightarrow{u} Y^{\bullet} \xrightarrow{v} Z^{\bullet} \xrightarrow{w} TX^{\bullet}$ in $\mathsf{K}^*(\mathcal{B})$, where $* =$ nothing, $+, -, b$.*

*Proof.* By Proposition 5.8. □

**Definition 6.13** (Homotopy Relation)**.** Two morphisms $f, g \in \mathrm{Hom}_{\mathsf{C}(\mathcal{B})}(X^{\bullet}, Y^{\bullet})$ is said to be *homotopic* (denote by $f \underset{h}{\simeq} g$) if there is a collection of morphisms $h = (h^n)$, $h^n : X^n \to Y^{n-1}$ such that

$$f^n - g^n = d_Y^{n-1} h^n + h^{n+1} d_X^n$$

for all $n \in \mathbb{Z}$. For $X^{\bullet}, Y^{\bullet} \in \mathsf{C}^*(\mathcal{B})$, $\mathrm{Htp}_{\mathsf{C}(\mathcal{B})}(X^{\bullet}, Y^{\bullet})$ is the subgroup of $\mathrm{Hom}_{\mathsf{C}(\mathcal{B})}(X^{\bullet}, Y^{\bullet})$ consisting of morphisms which are homotopic to 0.

**Proposition 6.14.** *For a morphism $f \in \mathrm{Hom}_{\mathsf{C}(\mathcal{B})}(X^{\bullet}, Y^{\bullet})$ ($* =$ nothing, $+, -, b$), the following are equivalent.*

1. *$f \in \mathrm{Htp}_{\mathsf{C}(\mathcal{B})}(X^{\bullet}, Y^{\bullet})$.*
2. *$f$ factors through $X^{\bullet} \xrightarrow{\mu_X} I^{\bullet}(X^{\bullet})$.*
3. *$f$ factors through $I^{\bullet}(T^{-1}Y^{\bullet}) \xrightarrow{\pi_{T^{-1}Y}} Y^{\bullet}$.*
4. *$f \in \mathcal{I}_{\mathsf{C}(\mathcal{B})}(X^{\bullet}, Y^{\bullet})$.*

*In particular,* $\mathrm{Htp}_{\mathsf{C}(\mathcal{B})}(X^{\bullet}, Y^{\bullet}) = \mathcal{I}_{\mathsf{C}(\mathcal{B})}(X^{\bullet}, Y^{\bullet})$.

*Proof.* 1 ⇔ 2. Let $h = (h^n)$ be a homotopy morphism $f \underset{h}{\simeq} 0$, and let $\phi = (\phi^n) : X^{\bullet} \to Y^{\bullet}$, $\phi^n = \left[\begin{smallmatrix} h^{n+1} & f^n \end{smallmatrix}\right]$. Then for all $n \in \mathbb{Z}$ we have

$$\begin{aligned} \phi^n d^n_{\mathrm{M}^{\bullet}(1_X)} &= \left[\begin{smallmatrix} h^{n+1} & f^n \end{smallmatrix}\right] \left[\begin{smallmatrix} -d_X^n & 0 \\ 1_{X^n} & d_X^{n-1} \end{smallmatrix}\right] \\ &= [-h^{n+1} d_X^n + f^n \;\; f^n d_X^{n-1}] \\ &= [d_Y^n h^n \;\; d_Y^{n-1} f^{n-1}] \\ &= d_Y^{n-1} \phi^{n-1}, \end{aligned}$$

$$\begin{aligned} \phi^n \mu_X^n &= \left[\begin{smallmatrix} h^{n+1} & f^n \end{smallmatrix}\right] \left[\begin{smallmatrix} 0 \\ 1_{X^n} \end{smallmatrix}\right] \\ &= f^n. \end{aligned}$$

Conversely, let $\phi = (\phi^n) : X^{\bullet} \to Y^{\bullet}$ be a morphism such that $f = \phi \mu_{X^{\bullet}}$. By the same calculation in the above, there is a homotopy morphism $h = (h^n)$, $h^n : X^n \to Y^{n+1}$ such that

$$f^n = d_Y^{n-1} h^n + h^{n+1} d_X^n$$

for all $n \in \mathbb{Z}$. Thus $f \in \mathrm{Htp}_{\mathsf{C}(\mathcal{B})}(X^{\bullet}, Y^{\bullet})$.

$2 \Leftrightarrow 4$. Since $I^{\cdot}(X^{\cdot})$ is $\mathcal{I}_{\mathsf{C}(\mathcal{B})}$-injective, $2 \Rightarrow 4$ is trivial. Conversely, since $\mu_{X^{\cdot}} : X^{\cdot} \to I^{\cdot}(X^{\cdot})$ is an admissible monomorphism, it is also trivial.

$3 \Leftrightarrow 4$. The same as $2 \Leftrightarrow 4$. □

**Corollary 6.15.** *The canonical functor $\mathsf{C}(\mathcal{B}) \to \mathsf{K}(\mathcal{B})$ preserves products and coproducts. In particular, we have*

1. *If $\mathcal{B}$ has products (resp., coproducts), then so has $\mathsf{K}(\mathcal{B})$.*
2. *If $\mathcal{A}$ satisfies the condition Ab3* (resp., Ab3), then $\mathsf{K}(\mathcal{A})$ has products (resp., coproducts) .*

*Proof.* By Proposition 6.14, we have exact sequences for $X^{\cdot}, Y^{\cdot} \in \mathsf{C}(\mathcal{B})$

$$\mathrm{Hom}_{\mathsf{C}(\mathcal{B})}(I^{\cdot}(X^{\cdot}), Y^{\cdot}) \to \mathrm{Hom}_{\mathsf{C}(\mathcal{B})}(X^{\cdot}, Y^{\cdot}) \to \mathrm{Hom}_{\mathsf{K}(\mathcal{B})}(X^{\cdot}, Y^{\cdot}) \to 0,$$
$$\mathrm{Hom}_{\mathsf{C}(\mathcal{B})}(X^{\cdot}, I^{\cdot}(T^{-1}Y^{\cdot})) \to \mathrm{Hom}_{\mathsf{C}(\mathcal{B})}(X^{\cdot}, Y^{\cdot}) \to \mathrm{Hom}_{\mathsf{K}(\mathcal{B})}(X^{\cdot}, Y^{\cdot}) \to 0.$$

Then it is easy. □

**Corollary 6.16.** *Let $\mathcal{B}'$ be another additive category, and $F : \mathsf{C}(\mathcal{B}) \to \mathsf{C}(\mathcal{B}')$ an additive functor. If $F$ satisfies the conditions*

(a) *there exists an functorial isomorphism $\alpha : FT_{\mathsf{C}(\mathcal{B})} \to T_{\mathsf{C}(\mathcal{B}')}F$,*
(b) *for any morphism $u : X^{\cdot} \to Y^{\cdot}$ in $\mathsf{C}(\mathcal{B})$, we have a commutative diagram*

$$\begin{array}{ccccccc}
FX^{\cdot} & \xrightarrow{Fu} & FY^{\cdot} & \xrightarrow{F\mu_u} & F\,\mathrm{M}^{\cdot}(u) & \xrightarrow{\alpha_X F\pi_u} & T_{\mathsf{C}(\mathcal{B}')}FX^{\cdot} \\
\| & & \| & & {\scriptstyle s}\downarrow & & \| \\
FX^{\cdot} & \xrightarrow{Fu} & FY^{\cdot} & \xrightarrow{\mu_{Fu}} & \mathrm{M}^{\cdot}(Fu) & \xrightarrow{\pi_{Fu}} & T_{\mathsf{C}(\mathcal{B}')}FX^{\cdot},
\end{array}$$

*then $F$ induces a $\partial$-functor $F' : \mathsf{K}(\mathcal{B}) \to \mathsf{K}(\mathcal{B}')$.*

**Remark 6.17.** By the proof of Proposition 6.14, $X^{\cdot}$ belongs to $\mathcal{I}_{\mathsf{C}(\mathcal{B})}$ if and only if $X^{\cdot}$ is a direct summand of $I^{\cdot}(X^{\cdot})$. Hence it is easy to see that any object of $\mathcal{I}_{\mathsf{C}(\mathcal{B})}$ is isomorphic to $I^{\cdot}(Z^{\cdot})$ for some $Z^{\cdot} \in \mathsf{C}(\mathcal{B})$.

**Exercise 6.18** (Biproduct)**.** Let $n \geq 0$, and let $X_i^{\cdot} : \ldots \to X_i^{n-1} \to X_i^n$ be complexes of $\mathsf{C}^-(\mathcal{B})$ indexed by $i \in \mathbb{N}$. Prove the following.

1. $\coprod_{i\in\mathbb{N}} T^i X_i^{\cdot} \cong \prod_{i\in\mathbb{N}} T^i X_i^{\cdot}$ in $\mathsf{C}(\mathcal{B})$. Thus $\bigoplus_{i\in\mathbb{N}} T^i X_i^{\cdot}$ exists in $\mathsf{C}(\mathcal{B})$.
2. $\coprod_{i\in\mathbb{N}} T^i X_i^{\cdot} \cong \prod_{i\in\mathbb{N}} T^i X_i^{\cdot}$ in $\mathsf{K}(\mathcal{B})$. Thus $\bigoplus_{i\in\mathbb{N}} T^i X_i^{\cdot}$ exists in $\mathsf{K}(\mathcal{B})$.

Let $n \geq 0$, and let $Y_i^{\cdot} : Y_i^0 \to Y_i^1 \to \ldots \to Y_i^n$ be complexes of $\mathsf{C}^{\mathrm{b}}(\mathcal{B})$ indexed by $i \in \mathbb{Z}$. Prove the following.

1. $\coprod_{i\in\mathbb{Z}} T^i Y_i^{\cdot} \cong \prod_{i\in\mathbb{Z}} T^i Y_i^{\cdot}$ in $\mathsf{C}(\mathcal{B})$. Thus $\bigoplus_{i\in\mathbb{Z}} T^i Y_i^{\cdot}$ exists in $\mathsf{C}(\mathcal{B})$.
2. $\coprod_{i\in\mathbb{Z}} T^i Y_i^{\cdot} \cong \prod_{i\in I} T^i Y_i^{\cdot}$ in $\mathsf{K}(\mathcal{B})$. Thus $\bigoplus_{i\in\mathbb{Z}} T^i Y_i^{\cdot}$ exists in $\mathsf{K}(\mathcal{B})$.

**Proposition 6.19.** *Let $R$ be a commutative complete local ring, $A$ a finite $R$-algebra, and $\mathcal{B}$ is a Krull-Schmidt subcategory of $\mathsf{mod}\,A$. Then $\mathsf{K}^{\mathrm{b}}(\mathcal{B})$ is also a Krull-Schmidt category.*

*Proof.* Let $X^{\cdot} \in \mathsf{C}^{\mathrm{b}}(\mathcal{B})$. Since we may assume that $X^{\cdot} = X^0 \to X^1 \to \ldots \to X^n$, $\mathrm{End}_{\mathsf{C}^{\mathrm{b}}(\mathcal{B})}(X^{\cdot})$ is a subring of $\prod_{i=0}^n \mathrm{End}_A(X^i)$, and hence $\mathrm{End}_{\mathsf{C}^{\mathrm{b}}(\mathcal{B})}(X^{\cdot})$ is semiperfect. It is clear that any idempotent of $\mathrm{End}_{\mathsf{C}^{\mathrm{b}}(\mathcal{B})}(X^{\cdot})$ splits. Therefore $\mathsf{C}^{\mathrm{b}}(\mathcal{B})$ is a Krull-Schmidt category. According to Theorem 3.13, we complete the proof. □

**Definition 6.20.** Let $\mathcal{A}$ be an abelian category. For a complex $X^{\cdot} = (X^n, d^n_X : X^n \to X^{n+1}_X)_{n\in\mathbb{Z}}$ of $\mathcal{A}$, we define an objects of $\mathcal{A}$ for all $n \in \mathbb{Z}$

$$\begin{aligned}\mathrm{Z}^n(X^{\cdot}) &= \operatorname{Ker} d^n_X\\ \mathrm{B}^n(X^{\cdot}) &= \operatorname{Im} d^{n-1}_X\\ \mathrm{C}^n(X^{\cdot}) &= \operatorname{Cok} d^{n-1}_X\\ \mathrm{H}^n(X^{\cdot}) &= \mathrm{Z}^n(X^{\cdot})/\,\mathrm{B}^n(X^{\cdot})\end{aligned}$$

this is called $n$th *cohomology*,

and define complexes

$$\begin{aligned}\mathrm{Z}^{\cdot}(X^{\cdot}) &= (\mathrm{Z}^n(X^{\cdot}), 0)_{n\in\mathbb{Z}}\\ \mathrm{B}^{\cdot}(X^{\cdot}) &= (\mathrm{B}^n(X^{\cdot}), 0)_{n\in\mathbb{Z}}\\ \mathrm{C}^{\cdot}(X^{\cdot}) &= (\mathrm{C}^n(X^{\cdot}), 0)_{n\in\mathbb{Z}}\\ \mathrm{H}^{\cdot}(X^{\cdot}) &= (\mathrm{H}^n(X^{\cdot}), 0)_{n\in\mathbb{Z}}.\end{aligned}$$

A complex $X^{\cdot} = (X^n, d^n_X)$ is called an *acyclic* complex if $\mathrm{H}^n(X^{\cdot}) = 0$ for all $n \in \mathbb{Z}$.

**Remark 6.21.** Since we have a commutative diagram

$$\begin{array}{ccccccccc} O & \longrightarrow & \mathrm{B}^n(X^{\cdot}) & \longrightarrow & X^n & \longrightarrow & \mathrm{C}^n(X^{\cdot}) & \longrightarrow & O\\ & & \downarrow & & \| & & \downarrow & & \\ O & \longrightarrow & \mathrm{Z}^n(X^{\cdot}) & \longrightarrow & X^n & \longrightarrow & \mathrm{B}^{n+1}(X^{\cdot}) & \longrightarrow & O,\end{array}$$

where all rows are exact, by snake lemma, we have a short exact sequence

$$O \to \mathrm{H}^n(X^{\cdot}) \to \mathrm{C}^n(X^{\cdot}) \to \mathrm{B}^{n+1}(X^{\cdot}) \to O.$$

**Exercise 6.22.** Let $\mathcal{A}$ be an abelian category, and $P$ a projective object of $\mathcal{A}$. For $X^{\cdot} \in \mathsf{C}(\mathcal{A})$, show that

$$\operatorname{Hom}_{\mathsf{K}(\mathcal{A})}(P, X^{\cdot}[i]) \cong \operatorname{Hom}_{\mathcal{A}}(P, \mathrm{H}^i(X^{\cdot}))$$

for all $i$.

**Proposition 6.23.** *Let $\mathcal{A}$ be an abelian category, and let $O \to X^{\cdot} \to Y^{\cdot} \to Z^{\cdot} \to O$ be exact in $\mathsf{C}(\mathcal{A})$. Then we have the induced long exact sequence*

$$\ldots \to \mathrm{H}^n(X^{\cdot}) \to \mathrm{H}^n(Y^{\cdot}) \to \mathrm{H}^n(Z^{\cdot}) \to \mathrm{H}^{n+1}(X^{\cdot}) \to \ldots$$

*Proof.* According to snake lemma, we have a commutative diagram

$$\begin{array}{ccccccccc} & & \mathrm{C}^n(X^{\cdot}) & \longrightarrow & \mathrm{C}^n(Y^{\cdot}) & \longrightarrow & \mathrm{C}^n(Z^{\cdot}) & \longrightarrow & O\\ & & \downarrow & & \downarrow & & \downarrow & & \\ O & \longrightarrow & \mathrm{Z}^{n+1}(X^{\cdot}) & \longrightarrow & \mathrm{Z}^{n+1}(Y^{\cdot}) & \longrightarrow & \mathrm{Z}^{n+1}(Z^{\cdot}) & & \end{array}$$

where all rows are exact. Then we get the exact sequence

$$\mathrm{H}^n(X^{\cdot}) \to \mathrm{H}^n(Y^{\cdot}) \to \mathrm{H}^n(Z^{\cdot}) \to \mathrm{H}^{n+1}(X^{\cdot}) \to \mathrm{H}^{n+1}(Y^{\cdot}) \to \mathrm{H}^{n+1}(Z^{\cdot}),$$

by snake lemma. □

**Remark 6.24.** Let $u$ be a morphism of $\operatorname{Hom}_{\mathsf{C}(\mathcal{A})}(X^{\cdot}, Y^{\cdot})$. According to Proposition 6.5, we have an exact sequence in $\mathsf{C}(\mathcal{A})$

$$O \to Y^{\cdot} \xrightarrow{\mu_u} \mathrm{M}^{\cdot}(u) \xrightarrow{\pi_u} TX^{\cdot} \to O.$$

Then we have the induced long exact sequence

$$\ldots \to \mathrm{H}^n(X^\bullet) \xrightarrow{\delta^n} \mathrm{H}^n(Y^\bullet) \to \mathrm{H}^n(\mathrm{M}^\bullet(u)) \to \mathrm{H}^{n+1}(X^\bullet) \xrightarrow{\delta^{n+1}} \ldots .$$

Moreover, it is not hard to see that $\delta^n = \mathrm{H}^n(u)$ for all $n \in \mathbb{Z}$ (cf. Proposition 10.4).

**Lemma 6.25.** *For $n \in \mathbb{Z}$, The functor $\mathrm{H}^n : \mathsf{C}(\mathcal{A}) \to \mathcal{A}$ factors through $\mathsf{K}(\mathcal{A})$.*

*Proof.* According to Remark 6.17, all objects of $\mathcal{I}_{\mathsf{C}(\mathcal{B})}$ are acyclic. Then by Proposition 6.14, it is trivial. $\square$

**Proposition 6.26.** *If $X^\bullet \xrightarrow{\underline{u}} Y^\bullet \xrightarrow{\underline{v}} Z^\bullet \xrightarrow{\underline{w}} TX^\bullet$ is a triangle in $\mathsf{K}(\mathcal{A})$, then we have the induced long exact sequence*

$$\ldots \to \mathrm{H}^n(X^\bullet) \xrightarrow{\mathrm{H}^n(\underline{u})} \mathrm{H}^n(Y^\bullet) \xrightarrow{\mathrm{H}^n(\underline{v})} \mathrm{H}^n(Z^\bullet) \xrightarrow{\mathrm{H}^n(\underline{w})} \mathrm{H}^{n+1}(X^\bullet) \to \ldots .$$

*Proof.* According to Remark 6.10, for a representative $u$ we have a commutative diagram

$$\begin{array}{ccccccccc}
O & \longrightarrow & X^\bullet & \longrightarrow & I^\bullet(X^\bullet) & \longrightarrow & TX^\bullet & \longrightarrow & O \\
 & & \downarrow{\scriptstyle u} & & \downarrow & & \| & & \\
O & \longrightarrow & Y^\bullet & \xrightarrow{v} & \mathrm{M}^\bullet(u) & \xrightarrow{w} & TX^\bullet & \longrightarrow & O
\end{array}$$

where all rows belong to $\mathcal{S}_{\mathsf{C}(\mathcal{A})}$, with $v = \mu_u$, $w = \pi_u$. By Proposition 6.23, we have the induced long exact sequence

$$\ldots \to \mathrm{H}^n(X^\bullet) \to \mathrm{H}^n(Y^\bullet) \to \mathrm{H}^n(\mathrm{M}^\bullet(u)) \to \mathrm{H}^{n+1}(X^\bullet) \to \ldots .$$

By Remark 6.24, Proposition 6.25, we get the statement. $\square$

## 7. Quotient Categories

**Definition 7.1** (Multiplicative System)**.** A *multiplicative system* in a category $\mathcal{C}$ is a collection $\mathsf{S}$ of morphisms in $\mathcal{C}$ which satisfies the following axioms:

(FR1) (1) $1_X \in \mathsf{S}$ for every $X \in \mathcal{C}$.
(2) For $s, t \in \mathsf{S}$, if $st$ is defined, then $st \in \mathsf{S}$.

(FR2) (1) Every diagram in $\mathcal{C}$

$$\begin{array}{ccc}
X & \xrightarrow{s} & Y \\
\downarrow{\scriptstyle f} & & \\
X' & &
\end{array}$$

with $s \in \mathsf{S}$, can be completed to a commutative square

$$\begin{array}{ccc}
X & \xrightarrow{s} & Y \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\
X' & \xrightarrow{t} & Y'
\end{array}$$

with $s, t \in \mathsf{S}$.
(2) Every diagram in $\mathcal{C}$

$$\begin{array}{ccc}
 & & Y \\
 & & \downarrow{\scriptstyle g} \\
X' & \xrightarrow{t} & Y'
\end{array}$$

with $t \in \mathsf{S}$, can be completed to a commutative square

$$\begin{array}{ccc} X & \xrightarrow{s} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow{t} & Y' \end{array}$$

with $s, t \in \mathsf{S}$.

(FR3) For $f, g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ the following are equivalent.
(1) There exists $s \in \mathsf{S}$ such that $sf = sg$.
(2) There exists $t \in \mathsf{S}$ such that $ft = gt$.

Throughout this section, $\mathsf{S}$ is a multiplicative system of a category $\mathcal{C}$.

**Definition 7.2** (Saturated Multiplicative System)**.** A multiplicative system $\mathsf{S}$ in a category $\mathcal{C}$ is called *saturated* if it satisfies the following axiom:

(FR0) For a morphism $s$ in $\mathcal{C}$, if there exist $f, g$ such that $sf, gs \in \mathsf{S}$, then $s \in \mathsf{S}$.

**Definition 7.3.** For a morphism $f : X \to Y$, we set $\mathrm{source}(f) = X$ and $\mathrm{sink}(f) = Y$.

For a multiplicative system $\mathsf{S}$, each $X \in \mathcal{C}$, $\mathsf{S}^X$ is a category defined by

1. $\mathrm{Ob}(\mathsf{S}^X) = \{s \in \mathsf{S} \mid \mathrm{source}(s) = X\}$,
2. $\mathrm{Hom}_{\mathsf{S}^X}(s, s') = \{f \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{sink}(s), \mathrm{sink}(s')) | s' = fs\}$ for $s, s' \in \mathrm{Ob}(\mathsf{S}^X)$,

and $\mathsf{S}_X$ is a category defined by

1. $\mathrm{Ob}(\mathsf{S}_X) = \{t \in \mathsf{S} \mid \mathrm{sink}(t) = X\}$,
2. $\mathrm{Hom}_{\mathsf{S}_X}(t, t') = \{f \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{source}(t), \mathrm{source}(t')) | t = t'f\}$ for $t, t' \in \mathrm{Ob}(\mathsf{S}_X)$.

**Lemma 7.4.** *For any $X \in \mathcal{C}$, $\mathsf{S}^X$ satisfies the following axioms:*

(L1) *For any $f_1 \in \mathrm{Hom}_{\mathsf{S}^X}(s, s_1{}')$, $f_2 \in \mathrm{Hom}_{\mathsf{S}^X}(s, s_2{}')$, there exist $s'' \in \mathsf{S}^X$ and $g_1 \in \mathrm{Hom}_{\mathsf{S}^X}(s_1{}', s'')$, $g_2 \in \mathrm{Hom}_{\mathsf{S}^X}(s_2{}', s'')$ such that $g_1 f_1 = g_2 f_2$.*
(L2) *For any $f_1, f_2 \in \mathrm{Hom}_{\mathsf{S}^X}(s, s')$, there exist $s'' \in \mathsf{S}^X$ and $g \in \mathrm{Hom}_{\mathsf{S}^X}(s', s'')$ such that $gf_1 = gf_2$.*
(L3) *$\mathsf{S}^X$ is connected.*

**Definition 7.5.** For $X, Y \in \mathcal{C}$, we define a covariant functor

$$h^X \circ \mathrm{sink} : \mathsf{S}^Y \to \mathfrak{Set}$$

where $h^X \circ \mathrm{sink}(s) = \mathrm{Hom}_{\mathcal{C}}(X, \mathrm{sink}(s))$ for $s \in \mathsf{S}^Y$, and a contravariant functor

$$h_Y \circ \mathrm{source} : \mathsf{S}_X \to \mathfrak{Set}$$

where $h_Y \circ \mathrm{source}(t) = \mathrm{Hom}_{\mathcal{C}}(\mathrm{source}(t), Y)$ for $t \in \mathsf{S}_X$.

**Lemma 7.6.** *Let $X, Y \in \mathcal{C}$. Define a relation , on the collection*

$$\{(f, s) | s \in \mathsf{S}^Y, f \in \mathrm{Hom}_{\mathcal{C}}(X, \mathrm{sink}(s))\}$$

*as follows: $(f_1, s_1) \sim (f_2, s_2)$ if and only if there exist $h_1 \in \mathrm{Hom}_{\mathsf{S}^Y}(s_1, s'), h_2 \in \mathrm{Hom}_{\mathsf{S}^Y}(s_2, s')$ such that $(h_1 f_1, s') = (h_2 f_2, s')$. Then $\sim$ is an equivalence relation and we have*

$$\operatorname*{\mathsf{colim}}_{\mathsf{S}^Y} h^X \circ \mathrm{sink} = \{(f, s) | s \in \mathsf{S}^Y, f \in \mathrm{Hom}_{\mathcal{C}}(X, \mathrm{sink}(s))\} / \sim .$$

*(Write the equivalence class $[(f, s)]$ for $(f, s)$, where $f \in \mathrm{Hom}_{\mathsf{C}}(X, \mathrm{sink}(s))$, $s \in \mathsf{S}^Y$.)*

**Remark 7.7** (Set-Theoretic Remark)**.** In the above, we dealt with $\mathsf{S}^Y$ as a small category (i.e. $\mathrm{Ob}(\mathsf{S}^Y)$ is a set). In general, we don't know the existence of the above colimit. But the above colimit exists if there is a small subcategory $\mathsf{S'}^Y$ of $\mathsf{S}^Y$ satisfying the following (this category is called a *cofinal subcategory*):

For any $Y \in \mathcal{C}$, $\mathsf{S}^Y$ satisfies the following axiom:
(Co) For any $s \in \mathsf{S}^Y$, there exists a morphism $f : s \to s'$ with $s' \in \mathsf{S'}^Y$.

Then $\operatorname*{colim}_{\mathsf{S'}^Y} h^X \circ \mathrm{sink}$ exists, and we have

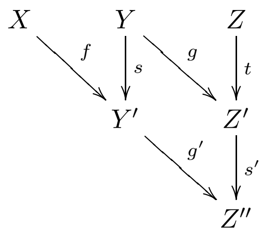
$$\operatorname*{colim}_{\mathsf{S}^Y} h^X \circ \mathrm{sink} = \operatorname*{colim}_{\mathsf{S'}^Y} h^X \circ \mathrm{sink}\,.$$

**Lemma 7.8.** *For any $X, Y, Z \in \mathcal{C}$ we have a well-defined mapping*

$$\operatorname*{colim}_{\mathsf{S}^Y} h^X \circ \mathrm{sink} \times \operatorname*{colim}_{\mathsf{S}^Z} h^Y \circ \mathrm{sink} \to \operatorname*{colim}_{\mathsf{S}^Z} h^X \circ \mathrm{sink}$$

*which is defined as follows: with each pair $([(f, s)], [(g, t)])$, since by (FR2) there exist $s' \in \mathsf{S}$ with $\mathrm{source}(s') = \mathrm{sink}(t)$, $g' \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{sink}(s), \mathrm{sink}(s'))$ such that $g's = s'g$, we associate the equivalence class $[(g'f, s't)]$.*

*Sketch.*




□

**Definition 7.9** (Quotient Category)**.** We define a category $\mathsf{S}^{-1}\mathcal{C}$, called the *quotient category* of $\mathcal{C}$, as follows:

1. $\mathrm{Ob}(\mathsf{S}^{-1}\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$.
2. For $X, Y \in \mathrm{Ob}(\mathcal{C})$, the morphism set is given by
$$\mathrm{Hom}_{\mathsf{S}^{-1}\mathcal{C}}(X, Y) = \operatorname*{colim}_{\mathsf{S}^Y} h^X \circ \mathrm{sink}\,.$$
3. For $X, Y, Z \in \mathrm{Ob}(\mathsf{S}^{-1}\mathcal{C})$, the law of composition is given by
$$\mathrm{Hom}_{\mathsf{S}^{-1}\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathsf{S}^{-1}\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathsf{S}^{-1}\mathcal{C}}(X, Z),$$
$$(([(f, s)], [(g, t)]) \mapsto [(g'f, s't)]),$$
where $[(g', s')] \in \mathrm{Hom}_{\mathsf{S}^{-1}\mathcal{C}}(\mathrm{sink}(s), \mathrm{sink}(t))$ with $g's = s'g$.
4. The identity of $X \in \mathrm{Ob}(\mathsf{S}^{-1}\mathcal{C})$ is given by the equivalence class $[(1_X, 1_X)]$.

**Definition 7.10** (Quotient Functor)**.** We have a functor $Q : \mathcal{C} \to \mathsf{S}^{-1}\mathcal{C}$, called the *quotient functor*, such that

$$Q(X) = X \text{ for } X \in \mathcal{C}, \quad Q(f) = [(f, 1_Y)] \text{ for } f \in \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

**Proposition 7.11** (Basic Properties)**.** *The following hold.*

1. *$Q : \mathcal{C} \to \mathsf{S}^{-1}\mathcal{C}$ sends null objects to null objects.*

2. *For* $f, g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ *the following are equivalent.*
   *(1)* $Q(f) = Q(g)$.
   *(2) There exists* $s \in \mathsf{S}^Y$ *such that* $sf = sg$.
   *(3) There exists* $t \in \mathsf{S}_X$ *such that* $ft = gt$.
3. *The following hold.*
   *(1)* $Q(s)$ *is an isomorphism for all* $s \in \mathsf{S}$.
   *(2) For any* $X, Y \in \mathsf{S}^{-1}\mathcal{C}$ *we have*

$$\begin{aligned}\mathrm{Hom}_{\mathsf{S}^{-1}\mathcal{C}}(X, Y) &= \{Q(s)^{-1}Q(f) \mid s \in \mathsf{S}^Y, f \in \mathrm{Hom}_{\mathcal{C}}(X, \mathrm{sink}(s))\}\\ &= \{Q(g)Q(t)^{-1} \mid t \in \mathsf{S}_X, g \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{source}(t), Y)\}.\end{aligned}$$

4. *For* $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ *the following are equivalent.*
   *(1)* $Q(f)$ *is an isomorphism.*
   *(2) There exist morphisms* $g, h$ *in* $\mathcal{C}$ *with* $gf, fh \in \mathsf{S}$.
5. *Assume* $\mathsf{S}$ *is saturated. Then for any* $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ *the following hold.*
   *(1)* $Q(f)$ *is an isomorphism if and only if* $f \in \mathsf{S}$.
   *(2) If there exists* $s \in \mathsf{S}^Y$ *with* $sf \in \mathsf{S}$, *then* $f \in \mathsf{S}$.
   *(3) If there exists* $t \in \mathsf{S}_X$ *with* $ft \in \mathsf{S}$, *then* $f \in \mathsf{S}$.
6. *For any* $X, Y \in \mathcal{C}$ *we have a bijection*

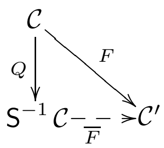
$$\zeta = \zeta_{X,Y} : \operatorname*{colim}_{\mathsf{S}^Y} h^X \circ \mathrm{sink} \to \operatorname*{colim}_{\mathsf{S}_X} h_Y \circ \mathrm{source}$$

*which associates with each* $[(f, s)]$ *the equivalence class of* $(t, g)$ *with* $ft = sg$, *and its inverse*

$$\eta = \eta_{X,Y} : \operatorname*{colim}_{\mathsf{S}_X} h_Y \circ \mathrm{source} \to \operatorname*{colim}_{\mathsf{S}^Y} h^X \circ \mathrm{sink}$$

*which associates with each* $[(t, g)]$ *the equivalence class of* $(f, s)$ *with* $ft = sg$.

**Proposition 7.12** (Uniqueness of Quotient)**.** *Let* $\mathcal{C}'$ *be another category and* $F : \mathcal{C} \to \mathcal{C}'$ *a functor such that* $F(s)$ *is an isomorphism for all* $s \in \mathsf{S}$. *Then there exists a unique functor* $\overline{F} : \mathsf{S}^{-1}\mathcal{C} \to \mathcal{C}'$ *such that* $F = \overline{F}Q$.




*Sketch.* Since every object of $\mathsf{S}^{-1}\mathcal{C}$ is of the form $QX$ for $X \in \mathcal{C}$, we can define $\overline{F} : \mathsf{S}^{-1}\mathcal{C} \to \mathcal{C}'$ as follows. Let $\overline{F}(QX) = F(X)$ for $QX \in \mathsf{S}^{-1}\mathcal{C}$ and $\overline{F}([(f, s)]) = (Fs)^{-1}Ff$ for $[(f, s)] \in \mathrm{Hom}_{\mathsf{S}^{-1}\mathcal{C}}(QX, QY)$. Then we have $F = \overline{F}Q$ and the required property. □

**Proposition 7.13.** *Let* $\mathcal{C}'$ *be another category and* $F, G : \mathsf{S}^{-1}\mathcal{C} \to \mathcal{C}'$ *functors. Then we have a bijective correspondence*

$$\mathrm{Mor}(F, G) \xrightarrow{\sim} \mathrm{Mor}(FQ, GQ), \quad (\zeta \mapsto \zeta_Q).$$

*Proof.* By the proof of the above lemma, it is easy. □

**Corollary 7.14.** *Let* $\tilde{\mathsf{S}} = \{f \mid Q(f)$ *is an isomorphism in* $\mathsf{S}^{-1}\mathcal{C}\}$. *Then* $\tilde{\mathsf{S}}$ *is a saturated multiplicative system, and* $\tilde{\mathsf{S}}^{-1}\mathcal{C}$ *is equivalent to* $\mathsf{S}^{-1}\mathcal{C}$.

**Remark 7.15.** Considering $\mathsf{S}^{-1}\mathcal{C}$, by Corollary 7.14, we may assume $\mathsf{S}$ is a saturated multiplicative system.

**Proposition 7.16.** *Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$. Assume $\mathsf{S} \cap \mathcal{D}$ is a multiplicative system in $\mathcal{D}$ and one of the following conditions is satisfied:*

(a) *For any $s \in \mathsf{S}^Y$ with $Y \in \mathcal{D}$, there exists $f \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{sink}(s), Y')$ with $Y' \in \mathcal{D}$ such that $fs \in \mathsf{S}$.*

(b) *For any $t \in \mathsf{S}_X$ with $X \in \mathcal{D}$, there exists $g \in \mathrm{Hom}_{\mathcal{C}}(X', \mathrm{source}(t))$ with $X' \in \mathcal{D}$ such that $tg \in \mathsf{S}$.*

*Then the canonical functor $(\mathsf{S} \cap \mathcal{D})^{-1}\mathcal{D} \to \mathsf{S}^{-1}\mathcal{C}$ is fully faithful, so that $(\mathsf{S} \cap \mathcal{D})^{-1}\mathcal{D}$ can be considered as a full subcategory of $\mathsf{S}^{-1}\mathcal{C}$.*

*Proof.* By the same reason of Remark 7.7. □

**Proposition 7.17.** *If $\mathcal{C}$ is an additive category, then $\mathsf{S}^{-1}\mathcal{C}$ is an additive category, and $Q : \mathcal{C} \to \mathsf{S}^{-1}\mathcal{C}$ is additive functor.*

*Proof.* It is easy to see that $\mathsf{S}^{-1}\mathcal{C}$ satisfies the condition of Definition 2.1. For $X = \coprod_{i=1}^{n} X_i$, by Proposition 2.3 we have $FX = \coprod_{i=1}^{n} FX_i$ in $\mathsf{S}^{-1}\mathcal{C}$. Therefore, $\mathsf{S}^{-1}\mathcal{C}$ is an additive category and $Q$ is an additive category by Definition 2.5, Proposition 2.10. □

## 8. Quotient Categories of Triangulated Categories

**Definition 8.1** (Compatible with Triangle)**.** A multiplicative system $\mathsf{S}$ in a triangulated category $\mathcal{C}$ is said to be *compatible with the triangulation* if it satisfies the following axioms:

(FR4) For a morphism $u$ in $\mathcal{C}$, $u \in \mathsf{S}$ if and only if $Tu \in \mathsf{S}$.

(FR5) For triangles $(X, Y, Z, u, v, w)$, $(X', Y', Z', u', v', w')$ and morphisms $f : X \to X'$, $g : Y \to Y'$ in $\mathsf{S}$ with $gu = u'f$, there exists $h : Z \to Z'$ in $\mathsf{S}$ such that $(f, g, h)$ is a homomorphism of triangles.

Throughout this section, $\mathcal{C}$ is a triangulated category, we assume that $\mathsf{S}$ is a saturated multiplicative system of $\mathcal{C}$ which is compatible with the triangulation, and $Q : \mathcal{C}' \to \mathsf{S}^{-1}\mathcal{C}$ is a quotient functor.

**Lemma 8.2.** *There exists a unique auto-functor $T_{\mathsf{S}^{-1}\mathcal{C}} : \mathsf{S}^{-1}\mathcal{C} \to \mathsf{S}^{-1}\mathcal{C}$ such that $QT_{\mathcal{C}} = T_{\mathsf{S}^{-1}\mathcal{C}}Q$.*

We simply write $T$ for $T_{\mathsf{S}^{-1}\mathcal{C}}$.

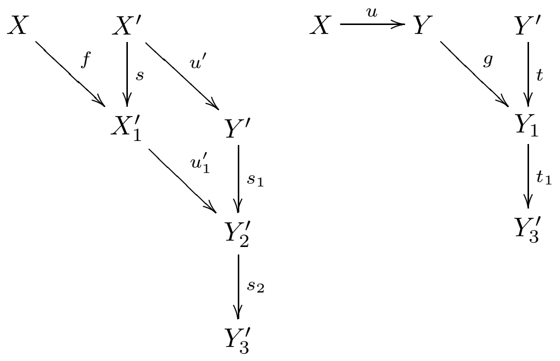
**Lemma 8.3.** *Let $(X, Y, Z, u, v, w)$, $(X', Y', Z', u', v', w')$ be triangles in $\mathcal{C}$, and let $\alpha \in \mathrm{Hom}_{\mathsf{S}^{-1}\mathcal{C}}(QX, QX')$, $\beta \in \mathrm{Hom}_{\mathsf{S}^{-1}\mathcal{C}}(QY, QY')$ such that $(Qu')\alpha = \beta Q(u)$. Then there exists $\gamma \in \mathrm{Hom}_{\mathsf{S}^{-1}\mathcal{C}}(QZ, QZ')$ such that*

$$\begin{array}{ccccccc}
QX & \xrightarrow{Qu} & QY & \xrightarrow{Qv} & QZ & \xrightarrow{Qw} & TQX \\
\alpha \downarrow & & \downarrow \beta & & \downarrow \gamma & & \downarrow T\alpha \\
QX' & \xrightarrow{Qu'} & QY' & \xrightarrow{Qv'} & QZ' & \xrightarrow{Qw'} & TQX'
\end{array}$$

*is commutative.*

*Proof.* There are $f : X \to X'_1$, $s : X' \to X'_1$, $g : Y \to Y'$, $t : Y' \to Y'_1$ such that $\alpha = (Qs)^{-1}Qf$, $\beta = (Qt)^{-1}Qg$ and $s, t \in \mathsf{S}$. Then there are $u'_1 : X'_1 \to Y'_2$,

$s_1 : Y' \to Y_2'$ such that $u_1' s = s_1 u'$, $s_1 \in \mathsf{S}$. Since $(Qu')\alpha = \beta Q(u)$, there are $s_2 : Y_2' \to Y_3'$, $t_1 : Y_1' \to Y_3'$ such that $s_2 u_1' f = t_1 g u$, $s_2 s_1 = t_1 t$ and $s_2 s_1 \in \mathsf{S}$.



Therefore there are $h : Z \to Z_1'$, $s_3 : Z' \to Z_1'$ such that we have a commutative diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle t_1 g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle Tf} \\
X_1 & \xrightarrow{s_2 u_1'} & Y_2' & \xrightarrow{v'} & Z_1' & \xrightarrow{w''} & TX_1' \\
{\scriptstyle s}\uparrow & & \uparrow{\scriptstyle s_2 s_1} & & \uparrow{\scriptstyle s_3} & & \uparrow{\scriptstyle Ts} \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & TX.
\end{array}$$

with $s_3 \in \mathsf{S}$. Since $(Qs_2 s_1)^{-1}(Qt_1)Qg = (Qt_1 t)^{-1}(Qt_1)Qg = (Qt)^{-1}Qg = \beta$, let $\gamma = (Qs_3)^{-1}h$, then $\gamma$ satisfies the statement. □

**Definition 8.4** (Triangulation)**.** A sextuple $(QX', QY', QZ', \lambda, \mu, \nu)$ in $\mathsf{S}^{-1}\mathcal{C}$ is called a triangle if there exists a triangle $(X, Y, Z, u, v, w)$ in $\mathcal{C}$ such that $(QX', QY', QZ', \lambda, \mu, \nu)$ is isomorphic to $(QX, QY, QZ, Qu, Qv, Qw)$.

**Theorem 8.5.** *$\mathsf{S}^{-1}\mathcal{C}$ is a triangulated category and $Q : \mathcal{C} \to \mathsf{S}^{-1}\mathcal{C}$ is a $\partial$-functor.*

*Proof.* Since for any morphism $\alpha : QX \to QY$ in $\mathsf{S}^{-1}\mathcal{C}$ there are $f : X \to Y_1$, $s : Y \to Y_1$, $t : X_1 \to Y$, $g : X_1 \to Y$ such that we have commutative diagram

$$\begin{array}{ccccc}
X & = & X & \xleftarrow{Qt} & X_1 \\
{\scriptstyle Qf}\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle Qg} \\
Y_1 & \xleftarrow{Qs} & Y & = & Y
\end{array}$$

with $s, t \in \mathsf{S}$, it is easy. □

**Proposition 8.6.** *Let $\mathcal{A}$ be an abelian category and $H : \mathcal{C} \to \mathcal{A}$ a cohomological functor such that $H(s)$ are isomorphisms for all $s \in \mathsf{S}$. Then there exists a unique cohomological functor $\overline{H} : \mathsf{S}^{-1}\mathcal{C} \to \mathcal{A}$ such that $H = \overline{H}Q$.*

**Proposition 8.7.** *Let $\mathcal{D}$ be another triangulated category and $F = (F, \theta) : \mathcal{C} \to \mathcal{D}$ a $\partial$-functor such that $Fs$ are isomorphisms for all $s \in \mathsf{S}$. Then there exists a unique*

*$\partial$-functor* $\overline{F} = (\overline{F}, \overline{\theta}) : \mathsf{S}^{-1}\mathcal{C} \to \mathcal{D}$ *such that* $F = \overline{F}Q$ *and* $\theta = \overline{\theta}Q$.

$$\begin{array}{ccc} \mathcal{C} & & \\ {\scriptstyle Q}\downarrow & \searrow^{F} & \\ \mathsf{S}^{-1}\mathcal{C} & \underset{\overline{F}}{\dashrightarrow} & \mathcal{C}' \end{array}$$

**Proposition 8.8.** *Let* $\mathcal{D}$ *be another triangulated category and* $F = (F, \theta), G = (G, \eta) : \mathsf{S}^{-1}\mathcal{C} \to \mathcal{D}$ *$\partial$-functors. Then we have a bijective correspondence*

$$\partial\operatorname{Mor}(F, G) \xrightarrow{\sim} \partial\operatorname{Mor}(FQ, GQ), \quad (\zeta \mapsto \zeta_Q).$$

*Proof.* By Proposition 7.13, it remains to check the following commutativity. For $X \in \mathcal{C}$, $\psi \in \partial\operatorname{Mor}(FQ, GQ)$, we have

$$\begin{array}{ccccc} FT_{\mathsf{S}^{-1}\mathcal{C}}QX & = & FQT_{\mathcal{C}}X & \xrightarrow{\theta Q} & T_{\mathcal{D}}FQX \\ & & {\scriptstyle \psi T_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle T_{\mathcal{D}}\psi} \\ GT_{\mathsf{S}^{-1}\mathcal{C}}QX & = & GQT_{\mathcal{C}}X & \xrightarrow[\eta Q]{} & T_{\mathcal{D}}GQX. \end{array}$$

□

## 9. Épaisse Subcategories

**Definition 9.1** (Épaisse Subcategory)**.** An *épaisse subcategory* $\mathcal{U}$ of a triangulated category $\mathcal{C}$ is a triangulated full subcategory of $\mathcal{C}$ such that if $u \in \operatorname{Hom}_{\mathcal{C}}(X, Y)$ factors through (an object of) $\mathcal{U}$ and is embedded in a triangle $(X, Y, Z, u, v, w)$ in $\mathcal{C}$ with $Z \in \mathcal{U}$, then $X, Y \in \mathcal{U}$.

**Proposition 9.2.** *For a triangulated full subcategory* $\mathcal{U}$ *of* $\mathcal{C}$*, the following are equivalent.*

1. $\mathcal{U}$ *is an épaisse subcategory of* $\mathcal{C}$*.*
2. $\mathcal{U}$ *is closed under direct summands.*

*Proof.* 1 ⇒ 2. Let $X, Y \in \mathcal{C}$ such that $X \oplus Y \in \mathcal{U}$. By Proposition 4.8, we have a triangle $(T^{-1}Y, X, X \oplus Y, 0, \mu, \pi)$. Since $0 : T^{-1}Y \to X$ factors through $O \in \mathcal{U}$, $T^{-1}Y, X \in \mathcal{U}$, and hence $Y, X \in \mathcal{U}$.

2 ⇒ 1. Let $(X, Y, Z, u, v, w)$ be a triangle such that $Z \in \mathcal{U}$ and $u$ factors through $Y' \in \mathcal{U}$, then we have a morphism of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{u'} & Y' & \longrightarrow & Z' & \longrightarrow & TX \\ \| & & \downarrow{\scriptstyle u''} & & \downarrow & & \| \\ X & \xrightarrow{u} & Y & \longrightarrow & Z & \longrightarrow & TX. \end{array}$$

By Proposition 4.8, we have a triangle $(Y', Z' \oplus Y, Z, *, *, *)$. We have $Z' \oplus Y \in \mathcal{U}$, and then $Y \in \mathcal{U}$. Therefore $Y, Z \in \mathcal{U}$ implies $X \in \mathcal{U}$. □

**Definition 9.3.** For an épaisse subcategory $\mathcal{U}$ of a triangulated category $\mathcal{C}$, we denote by $\Phi(\mathcal{U})$ the collection of morphisms $u$ in $\mathcal{U}$. such that $\mathrm{M}(u) \in \mathcal{U}$.

**Lemma 9.4.** *Let* $\mathcal{U}$ *be an épaisse subcategory of a triangulated category* $\mathcal{C}$*. For a morphism* $f$ *in* $\mathcal{C}$*, the following are equivalent.*

1. $f$ *factors through some object of* $\mathcal{U}$*.*

2. *There exists* $s \in \Phi(\mathcal{U})$ *such that* $sf = 0$.
3. *There exists* $t \in \Phi(\mathcal{U})$ *such that* $ft = 0$.

*Proof.* 1 $\Leftrightarrow$ 2. If $f$ factors through $U \in \mathcal{U}$, then we have a commutative diagram

$$\begin{array}{ccccccc} X & = & X & & & & \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle f} & & & & \\ U & \xrightarrow{x} & Y & \xrightarrow{s} & Z & \xrightarrow{z} & TU. \end{array}$$

where the bottom row is a triangle, and with $s \in \Phi(\mathcal{U})$. We have $sf = 0$. Conversely, if $sf = 0$ with $s \in \Phi(\mathcal{U})$, then there exists $u$ such that we also have the above commutative diagram. Therefore $f = xu$.

1 $\Leftrightarrow$3. Similarly. ☐

**Proposition 9.5.** *Let* $\mathcal{U}$ *be an épaisse subcategory of a triangulated category* $\mathcal{C}$. *Then* $\Phi(\mathcal{U})$ *is a saturated multiplicative system which is compatible with the triangulation.*

*Proof.* We use the following diagram to check the axioms of a multiplicative system.

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{i} & Z' & \xrightarrow{i'} & TX \\ \| & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle z} & & \| \\ X & \xrightarrow{w} & Z & \xrightarrow{k} & Y' & \xrightarrow{k'} & TX \\ & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle Tu} \\ & & X' & = & X' & \xrightarrow{j'} & TY \\ & & \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle x} & & \\ & & TY & \xrightarrow{Ti} & TZ' & & \end{array}$$

Diagram A

(FR0) Let $v : X \to Y, u : Y \to Z, r : Z \to U$ be morphisms such that $ru, uv \in \Phi(\mathcal{U})$. Then we have a commutative diagram

$$\begin{array}{ccccccc} Y & \xrightarrow{u} & Z & \xrightarrow{j} & X' & \xrightarrow{j'} & TY \\ \| & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle l} & & \| \\ Y & \xrightarrow{ru} & U & \xrightarrow{p} & V & \xrightarrow{q} & TY \end{array}$$

with $V \in \mathcal{U}$. Since $x = (Ti)j' = (Ti)ql$, $x$ factors through $V \in \mathcal{U}$. Since $uv \in \Phi(\mathcal{U})$, we have Diagram A with $Y' \in \mathcal{U}$. Therefore, $X', Z' \in \mathcal{U}$, and hence $u \in \Phi(\mathcal{U})$.

(FR1) (1) Since $O \in \mathcal{U}$, it is trivial. (2) If $u, v \in \Phi(\mathcal{U})$, then we have Diagram A with $Z', X' \in \mathcal{U}$. Then $Y' \in \mathcal{U}$, and hence $vu \in \Phi(\mathcal{U})$.

(FR2) (1) Given $v \in\in \Phi(\mathcal{U})$, $i$, we have Diagram A with $X' \in \mathcal{U}$. Then $z \in \Phi(\mathcal{U})$. (2) Given $z \in \Phi(\mathcal{U})$, $k$, we have Diagram A with $X' \in \mathcal{U}$. Then $v \in \Phi(\mathcal{U})$.

(FR3) By Lemma 9.4.

(FR4) It is trivial.

(FR5) By Proposition 4.9, it is easy. ☐

**Theorem 9.6.** *For a saturated multiplicative system* $\mathsf{S}$ *of a triangulated category* $\mathcal{C}$ *which is compatible with the triangulation, let* $\Psi(\mathsf{S})$ *be the full subcategory of* $\mathcal{C}$ *consisting of objects* $X$ *such that* $QX = O$*, where* $Q : \mathcal{C} \to \mathsf{S}^{-1}\mathcal{C}$ *is the canonical quotient. Then* $\Psi(\mathsf{S})$ *is an épaisse subcategory of* $\mathcal{C}$*.*

*Hence,* $\Phi$ *and* $\Psi$ *induce one to one correspondence between épaisse subcategories and saturated multiplicative systems which is compatible with the triangulation.*

*Proof.* Let $(QX, QY, QZ, Qu, Qv, Qw)$ be the image of a triangle $(X, Y, Z, u, v, w)$ of $\mathcal{C}$. If two objects of $QX, QY, QZ$ are $O$, then the rest one is clearly $O$. Then it is easy to see that $\Psi(\mathsf{S})$ is a triangulated full subcategory of $\mathcal{C}$. If $Z \in \Psi(\mathsf{S})$ and $u$ factors through some object of $\Psi(\mathsf{S})$, then $QZ = O$ and $Qu$ factors through $O$. Therefore, $QX = QY = O$ and hence $X, Y \in \Psi(\mathsf{S})$. □

**Definition 9.7.** For an épaisse subcategory $\mathcal{U}$ of a triangulated category $\mathcal{C}$, we denote by $\mathcal{C}/\mathcal{U}$ the quotient category $\Phi(\mathcal{U})^{-1}\mathcal{C}$. In this case, we say that $0 \to \mathcal{U} \to \mathcal{C} \to \mathcal{C}/\mathcal{U} \to 0$ is an exact sequence of triangulated categories.

**Proposition 9.8.** *Let* $\mathcal{C}$ *be a triangulated category,* $\mathcal{U}$ *an épaisse subcategory of* $\mathcal{C}$*, and* $Q : \mathcal{C} \to \mathcal{C}/\mathcal{U}$ *the canonical quotient. For* $M \in \mathcal{C}$*, the following are equivalent.*

1. *For every* $f : X \to Y \in \Phi(\mathcal{U})$*,* $\mathrm{Hom}_{\mathcal{C}}(f, M) : \mathrm{Hom}_{\mathcal{C}}(Y, M) \to \mathrm{Hom}_{\mathcal{C}}(X, M)$ *is bijective.*
2. $\mathrm{Hom}_{\mathcal{C}}(\mathcal{U}, M) = 0$.
3. *For every* $X \in \mathcal{C}$*,* $Q_{X,M} : \mathrm{Hom}_{\mathcal{C}}(X, M) \to \mathrm{Hom}_{\mathcal{C}/\mathcal{U}}(QX, QM)$ *is bijective.*

*Proof.* $1 \Rightarrow 2$. For every object $U \in \mathcal{U}$, $O \to U \xrightarrow{1} U \to O$ is a triangle. Then $0 = \mathrm{Hom}_{\mathcal{C}}(O, M) \cong \mathrm{Hom}_{\mathcal{C}}(U, M)$.

$2 \Rightarrow 3$. Every morphism of $\mathrm{Hom}_{\mathcal{C}/\mathcal{U}}(QX, QM)$ is represented by a diagram

$$\begin{array}{ccc} X' & & \\ \downarrow{\scriptstyle s} & \searrow^{f} & \\ X & & M \end{array}$$

where $s$ is contained in a triangle $U \to X' \xrightarrow{s} X \to TU$ with $U \in \mathcal{U}$. Then there exists $f' : X \to M$ in $\mathcal{C}$ such that $f = f's$, because $\mathrm{Hom}_{\mathcal{C}}(U, M) = 0$. Hence $Q_{X,M}$ is surjective. Let $U \to X' \to X \to TU$ be a triangle with $U \in \mathcal{U}$. If a morphism $g : X \to M$ satisfies $gs = 0$, then there exist $u : TU \to M$ such that $g = ut$. Therefore $g = 0$, because $u \in \mathrm{Hom}_{\mathcal{C}}(U, M) = 0$. Hence $Q_{X,M}$ is injective.

$3 \Rightarrow 1$. Let $f : X \to Y$ be a morphism in $\Phi(\mathcal{U})$. Then we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(Y, M) & \xrightarrow{\mathrm{Hom}(f,M)} & \mathrm{Hom}_{\mathcal{C}}(X, M) \\ {\scriptstyle Q_{Y,M}}\downarrow & & \downarrow{\scriptstyle Q_{X,M}} \\ \mathrm{Hom}_{\mathcal{C}/\mathcal{U}}(QY, QM) & \xrightarrow{\mathrm{Hom}(Qf,QM)} & \mathrm{Hom}_{\mathcal{C}/\mathcal{U}}(QX, QM) \end{array}$$

According to 3, $Q_{X,M}$ and $Q_{Y,M}$ are bijective. Since $Q\mathcal{U} = 0$, $\mathrm{Hom}(Qf, QM)$ is bijective. Hence $\mathrm{Hom}(f, M)$ is bijective. □

**Definition 9.9** ($\mathcal{U}$-Local Object)**.** An object $M$ is called $\mathcal{U}$*-local* (resp., $\mathcal{U}$*-colocal*) if it satisfies the equivalent conditions (resp., the dual conditions ) of Proposition 9.8 . Let $0 \to \mathcal{U} \to \mathcal{C} \xrightarrow{Q} \mathcal{C}/\mathcal{U} \to 0$ be an exact sequence of triangulated categories.

The right (resp., left ) adjoint of $Q$ is called a *section functor*. If there exists a section functor $S$, then $\{\mathcal{C}/\mathcal{U}; Q, S\}$ is called a *localization* (resp., *colocalization*) of $\mathcal{C}$, and $0 \to \mathcal{U} \to \mathcal{C} \xrightarrow{Q} \mathcal{C}/\mathcal{U} \to 0$ is called *localization* (resp., *colocalization*) *exact.*

**Lemma 9.10.** *Let $\{\mathcal{C}/\mathcal{U}; Q, S\}$ be a localization of $\mathcal{C}$. For every object $V \in \mathcal{C}/\mathcal{U}$, $SV$ is $\mathcal{U}$-local.*

*Proof.* For every $f : X \to Y \in \Phi(\mathcal{U})$, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(Y, SV) & \xrightarrow{\mathrm{Hom}(f,SV)} & \mathrm{Hom}_{\mathcal{C}}(X, SV) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{Hom}_{\mathcal{C}/\mathcal{U}}(QY, V) & \xrightarrow{\mathrm{Hom}(Qf,V)} & \mathrm{Hom}_{\mathcal{C}/\mathcal{U}}(QX, V) \end{array}$$

Therefore $\mathrm{Hom}(f, SV)$ is an isomorphism. By Proposition 9.8, $SV$ is $\mathcal{U}$-local. □

**Proposition 9.11.** *Let $\{\mathcal{C}/\mathcal{U}; Q, S\}$ be a localization of $\mathcal{C}$, and $\tau : QS \to \mathbf{1}_{\mathcal{C}/\mathcal{U}}$ and $\sigma : \mathbf{1}_{\mathcal{C}} \to SQ$ adjunction arrows. Then the following hold.*

1. *$\tau$ is an isomorphism (i.e. $S$ is fully faithful ).*
2. *For every object $X \in \mathcal{C}$, the triangle $U \to X \xrightarrow{\sigma_X} SQX \to TU$ satisfies that $U$ is in $\mathcal{U}$.*

*Proof.* 1. For every $X \in \mathcal{C}$ and $Y \in \mathcal{C}/\mathcal{U}$, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X, SY) & = & \mathrm{Hom}_{\mathcal{C}}(X, SY) \\ {\scriptstyle Q_{X,SY}} \downarrow & & \downarrow \wr \\ \mathrm{Hom}_{\mathcal{C}/\mathcal{U}}(QX, QSY) & \xrightarrow{\mathrm{Hom}(QX,\tau_Y)} & \mathrm{Hom}_{\mathcal{C}/\mathcal{U}}(QX, Y). \end{array}$$

By Proposition 9.8 and Lemma 9.10, $Q_{X,SY}$ is an isomorphism. Then $\mathrm{Hom}(QX, \tau_Y)$ is an isomorphism. For any $Z \in \mathcal{C}/\mathcal{U}$, there exists $X \in \mathcal{C}$ such that $Z \cong QX$. Hence $\tau$ is an isomorphism.

2. It suffices to show that for any $X \in \mathcal{C}$, $Q\sigma_X$ is an isomorphism. By the property of adjunction arrows, we have $QX \xrightarrow{Q\sigma_X} QSQX \xrightarrow{\tau_{QX}} QX = \mathbf{1}_{QX}$, and hence $Q\sigma_X$ is an isomorphism. □

**Corollary 9.12.** *Let $M \in \mathcal{C}$. Then $M$ is $\mathcal{U}$-local if and only if $M \cong SQM$.*

**Proposition 9.13.** *Let $\mathcal{C}$ and $\mathcal{C}'$ be triangulated categories, $F : \mathcal{C} \to \mathcal{C}'$ a $\partial$- functor which has a fully faithful right adjoint $S : \mathcal{C}' \to \mathcal{C}$. Then $F$ induces an equivalence between $\mathcal{C}/\operatorname{Ker} F$ and $\mathcal{C}'$.*

*Proof.* By the universal property of $Q : \mathcal{C} \to \mathcal{C}/\operatorname{Ker} F$, we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{C} & & \\ {\scriptstyle Q}\downarrow & \searrow^{F} & \\ \mathcal{C}/\operatorname{Ker} F & \xrightarrow[F']{} & \mathcal{C}' \end{array}$$

If $f : X \to Y$ is a morphism in $\mathcal{C}$, then $Ff$ is an isomorphism if and only if $Qf$ is an isomorphism. For every object $M \in \mathcal{C}$, $FM \to FSFM$ is an isomorphism, and then $QM \to QSFM$ is an isomorphism. Therefore $Q \to QSF$ is an isomorphism.

By the universal property of $Q$ and $QSF = QSF'Q$, we have $\mathbf{1}_{\mathcal{C}/\operatorname{Ker} F} \cong QSF'$. Since, $F'QS = FS \cong \mathbf{1}_{\mathcal{C}'}$, $F'$ is an equivalence. □

**Definition 9.14** (stable $t$-structure)**.** For full subcategories $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{C}$, $(\mathcal{U}, \mathcal{V})$ is called a *stable $t$-structure* in $\mathcal{C}$ provided that

1. $\mathcal{U}$ and $\mathcal{V}$ are stable for translations.
2. $\operatorname{Hom}_{\mathcal{C}}(\mathcal{U}, \mathcal{V}) = 0$.
3. For every $X \in \mathcal{C}$, there exists a triangle $U \to X \to V \to TU$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

**Proposition 9.15.** *Let $\mathcal{C}$ be a triangulated category, $(\mathcal{U}, \mathcal{V})$ a stable $t$-structure in $\mathcal{C}$. Then the following hold.*

1. *For $X \in \mathcal{C}$, $\operatorname{Hom}_{\mathcal{C}}(X, \mathcal{V}) = 0$ if and only if $X$ is isomorphic to an object of $\mathcal{U}$.*
2. *For $Y \in \mathcal{C}$, $\operatorname{Hom}_{\mathcal{C}}(\mathcal{U}, Y) = 0$ if and only if $Y$ is isomorphic to an object of $\mathcal{V}$.*
3. *Let $\tilde{\mathcal{U}}$ be the full subcategory of $\mathcal{C}$ consisting objects which are isomorphic to objects of $\mathcal{U}$. Then $\tilde{\mathcal{U}}$ is an épaisse subcategory of $\mathcal{C}$.*
4. *Let $\tilde{\mathcal{V}}$ be the full subcategory of $\mathcal{C}$ consisting objects which are isomorphic to objects of $\mathcal{V}$. Then $\tilde{\mathcal{V}}$ is an épaisse subcategory of $\mathcal{C}$.*

*Proof.* 1. For $X \in \mathcal{C}$, we have a triangle

$$U_X \xrightarrow{\tau_X} X \xrightarrow{\sigma_X} V_X \to TU_X.$$

If $\operatorname{Hom}_{\mathcal{C}}(X, \mathcal{V}) = 0$, then $\sigma_X = 0$ and $U_X \cong X \oplus T^{-1}V_X$. Therefore, $T^{-1}V_X = O$ and $X \cong U_X$, because of $\operatorname{Hom}_{\mathcal{C}}(U_X, T^{-1}V_X) = 0$.

2. Similarly.

3, 4. By 1, 2, it is trivial. □

**Proposition 9.16.** *Let $\mathcal{C}$ be a triangulated category. If $\{\mathcal{V}; Q, S\}$ is a localization of $\mathcal{C}$, then $S$ is fully faithful, $(\mathcal{U}, S\mathcal{V})$ is a stable $t$-structure, where $\mathcal{U} = \operatorname{Ker} Q$. Conversely, if $(\mathcal{U}, \mathcal{V})$ is a stable $t$-structure in $\mathcal{C}$, then the canonical inclusion $S : \mathcal{V} \to \mathcal{C}$ has a left adjoint $Q$ such that $\{\mathcal{V}; Q, S\}$ is a localization.*

*Proof.* Let $\{\mathcal{V}; Q, S\}$ be a localization of $\mathcal{C}$. Then, by

$$\begin{aligned} \operatorname{Hom}_{\mathcal{C}}(\mathcal{U}, S\mathcal{V}) &\cong \operatorname{Hom}_{\mathcal{C}}(Q\mathcal{U}, \mathcal{V}) \\ &= 0 \end{aligned}$$

and Proposition 9.13, it is clear that $S$ is fully faithful and $(\mathcal{U}, S\mathcal{V})$ is a stable $t$-structure. Conversely, let $(\mathcal{U}, \mathcal{V})$ be a stable $t$-structure in $\mathcal{C}$. For $X \in \mathcal{C}$, let $U_X \to X \to V_X \to TU_X$ be triangle such that $U_X \in \mathcal{U}$ and $V_X \in \mathcal{V}$. Since $\operatorname{Hom}_{\mathcal{C}}(\mathcal{U}, \mathcal{V}) = 0$ and $\mathcal{U}$ and $\mathcal{V}$ are stable under translations, for any $V \in \mathcal{V}$, we have an isomorphism

$$\operatorname{Hom}_{\mathcal{C}}(V_X, V) \cong \operatorname{Hom}_{\mathcal{C}}(X, V).$$

According to Theorem 1.18, $S : \mathcal{V} \to \mathcal{C}$ has a left adjoint $Q$ such that $\{\mathcal{V}; Q, S\}$ is a localization. □

**Remark 9.17.** Similarly, if $(\mathcal{U}, \mathcal{V})$ is a stable $t$-structure in $\mathcal{C}$, then there is a functor $Q : \mathcal{C} \to \mathcal{U}$ such that $\{\mathcal{U}; Q, S'\}$ is a colocalization of $\mathcal{C}$, where $S' : \mathcal{U} \to \mathcal{C}$ is the canonical embedding. Conversely, if $\{\mathcal{U}; Q, S'\}$ is a colocalization of $\mathcal{C}$, then $(S'\mathcal{U}, \operatorname{Ker} Q)$ is a stable $t$-structure in $\mathcal{C}$.

## 10. Derived Categories

Throughout this section, $\mathcal{A}$ is an abelian category.

**Definition 10.1.** For $X^\cdot, Y^\cdot \in \mathsf{K}^*(\mathcal{A})$, a morphism $u \in \mathrm{Hom}_{\mathsf{C}(\mathcal{A})}(X^\cdot, Y^\cdot)$ is called a *quasi-isomorphism* if $\mathrm{H}^n(u)$ are isomorphisms for all $n \in \mathbb{Z}$, where $* =$ nothing, $+, -, b$.

$\mathsf{K}^{*,\phi}(\mathcal{A})$ is a full subcategory of $\mathsf{K}^*(\mathcal{A})$ consisting of complexes of which all homologies are $O$, where $* =$ nothing, $+, -, b$.

*Proof.* It is easy to see that $\mathsf{K}^{*,\phi}(\mathcal{A})$ is an épaisse subcategory of $\mathsf{K}^*(\mathcal{A})$. By Proposition 6.26, it is easy. □

**Definition 10.2** (Derived Category)**.** The derived category $\mathsf{D}^*(\mathcal{A})$ of an abelian category $\mathcal{A}$ is $\mathsf{K}^*(\mathcal{A})/\mathsf{K}^{*,\phi}(\mathcal{A})$, where $* =$ nothing, $+, -, b$.

**Remark 10.3.** For two morphisms $f, g : X^\cdot \to Y^\cdot$ in $\mathsf{C}(\mathcal{A})$, $f = g \Rightarrow f \underset{h}{\simeq} g \Rightarrow f \cong g$ in $\mathsf{D}(\mathcal{A}) \Rightarrow \mathrm{H}^n(f) \cong \mathrm{H}^n(g)$ for all $n$. The converse implications do not hold.

**Proposition 10.4.** *If $O \to X^\cdot \xrightarrow{u} Y^\cdot \xrightarrow{v} Z^\cdot \to 0$ is a exact sequence in $\mathsf{C}(\mathcal{A})$, then it can be embedded in a triangle in $\mathsf{D}(\mathcal{A})$*

$$QX^\cdot \xrightarrow{Qu} Y^\cdot \xrightarrow{Qv} QZ^\cdot \xrightarrow{Qw} TQX^\cdot.$$

*Proof.* According to Remark 6.10, we have a commutative diagram in $\mathsf{C}(\mathcal{A})$

$$\begin{array}{ccccccccc}
 & & O & & O & & & & \\
 & & \downarrow & & \downarrow & & & & \\
O & \longrightarrow & X^\cdot & \longrightarrow & I^\cdot(X^\cdot) & \longrightarrow & TX^\cdot & \longrightarrow & O \\
 & & {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle x} & & \| & & \\
O & \longrightarrow & Y^\cdot & \xrightarrow{v'} & \mathrm{M}^\cdot(u) & \xrightarrow{w} & TX^\cdot & \longrightarrow & O \\
 & & {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle s} & & & & \\
 & & Z^\cdot & = & Z^\cdot & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & O & & O & & & &
\end{array}$$

where all rows and columns are exact. Then $X^\cdot \xrightarrow{u} Y^\cdot \xrightarrow{v'} Z^\cdot \xrightarrow{w} TX^\cdot$ is a triangle in $\mathsf{K}(\mathcal{A})$. Since $I^\cdot(X^\cdot) \in \mathsf{K}^\phi(\mathcal{A})$, by Proposition 6.23, $s$ is a quasi-morphism, and hence we have a commutative diagram in $\mathsf{D}(\mathcal{A})$

$$\begin{array}{ccccccc}
QX^\cdot & \xrightarrow{Qu} & QY^\cdot & \xrightarrow{Qv'} & Q\,\mathrm{M}^\cdot(u) & \xrightarrow{Qw} & TQX^\cdot \\
\| & & \| & & \downarrow{\scriptstyle Qs} & & \| \\
QX^\cdot & \xrightarrow{Qu} & QY^\cdot & \xrightarrow{Qv} & QZ & \xrightarrow{Qw(Qs)^{-1}} & TQX^\cdot.
\end{array}$$

□

**Definition 10.5.** A full subcategory $\mathcal{A}'$ of $\mathcal{A}$ is called a *thick abelian* full subcategory if $\mathcal{A}'$ is an abelian exact full subcategory which is closed under extensions.

For $* =$ nothing, $+, -, b$, we denote by $\mathsf{K}^*_{\mathcal{A}'}(\mathcal{A})$ the full subcategory of $\mathsf{K}^*(\mathcal{A})$ consisting of complexes $X^\cdot \in \mathsf{K}(\mathcal{A})$ with $\mathrm{H}^n(X^\cdot) \in \mathcal{A}'$ for all $n \in \mathbb{Z}$.

Moreover, we set $\mathsf{D}^*_{\mathcal{A}'}(\mathcal{A}) = \mathsf{K}^*_{\mathcal{A}'}(\mathcal{A}) / \mathsf{K}^{*,\phi}(\mathcal{A})$, where $* =$ nothing, $+, -, b$.

**Definition 10.6** (Truncations)**.** For a complex $X^\cdot = (X^i, d^i)$, we define the following truncations:

$$\begin{aligned}
\sigma_{>n}X^\cdot &: \dots \to 0 \to \operatorname{Im} d^n \to X^{n+1} \to X^{n+2} \to \dots, \\
\sigma_{\leq n}X^\cdot &: \dots \to X^{n-2} \to X^{n-1} \to \operatorname{Ker} d^n \to 0 \to \dots, \\
\sigma'_{\geq n}X^\cdot &: \dots \to 0 \to \operatorname{Cok} d^{n-1} \to X^{n+1} \to X^{n+2} \to \dots, \\
\sigma'_{<n}X^\cdot &: \dots \to X^{n-2} \to X^{n-1} \to \operatorname{Im} d^{n-1} \to 0 \to \dots, \\
\tau_{\geq n}X^\cdot &: \dots \to 0 \to X^n \to X^{n+1} \to X^{n+2} \to \dots, \\
\tau_{\leq n}X^\cdot &: \dots \to X^{n-2} \to X^{n-1} \to X^n \to 0 \to \dots.
\end{aligned}$$

Then we have exact sequences in $\mathsf{C}(\mathcal{A})$

$$\begin{gathered}
O \to \sigma_{\leq n}(X^\cdot) \to X^\cdot \to \sigma_{>n}(X^\cdot) \to O \\
O \to \sigma'_{<n}(X^\cdot) \to X^\cdot \to \sigma'_{\geq n}(X^\cdot) \to O \\
O \to \tau_{\geq n}(X^\cdot) \to X^\cdot \to \tau_{\leq n+1}(X^\cdot) \to O
\end{gathered}$$

Then it is easy to see that

$$\begin{aligned}
\mathrm{H}^i(\sigma_{>n}X^\cdot) &= \begin{cases} O & \text{if } i \leq n \\ \mathrm{H}^i(X^\cdot) & \text{if } i > n \end{cases} \\
\mathrm{H}^i(\sigma'_{\geq n}X^\cdot) &= \begin{cases} O & \text{if } i < n \\ \mathrm{H}^i(X^\cdot) & \text{if } i \geq n \end{cases} \\
\mathrm{H}^i(\sigma_{\leq n}X^\cdot) &= \begin{cases} \mathrm{H}^i(X^\cdot) & \text{if } i \leq n \\ O & \text{if } i > n \end{cases} \\
\mathrm{H}^i(\sigma'_{<n}X^\cdot) &= \begin{cases} \mathrm{H}^i(X^\cdot) & \text{if } i < n \\ O & \text{if } i \geq n \end{cases}
\end{aligned}$$

**Proposition 10.7.** *The following hold.*

1. *The canonical functor* $\mathsf{D}^*(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ *is fully faithful, where* $* = +, -$.
2. *The canonical functor* $\mathsf{D}^{\mathrm{b}}(\mathcal{A}) \to \mathsf{D}^*(\mathcal{A})$ *is fully faithful, where* $* = +, -$.
3. *The canonical functor* $\mathsf{D}^*_{\mathcal{A}'}(\mathcal{A}) \to \mathsf{D}^*(\mathcal{A})$ *is fully faithful, where* $* = +, -, b$.

*Proof.* According to Definitions 10.2, 10.5, it suffices to check the condition of Proposition 7.16. Let $X^\cdot \in \mathsf{K}^-(\mathcal{A})$, $Y^\cdot \in \mathsf{K}(\mathcal{A})$ and a quasi-isomorphism $X^\cdot \to Y^\cdot$ in $\mathsf{K}(\mathcal{A})$. Then we may assume that there is $n$ such that $\mathrm{H}^i(Y^\cdot) = 0$ for all $i > n$. Then the morphism $Y^\cdot \to \sigma_{\leq n}(Y^\cdot)$ is a quasi-isomorphism, and $\sigma_{\leq n}(Y^\cdot) \in \mathsf{K}^-(\mathcal{A})$. For the other cases, similarly. □

Let $\mathsf{Inj}\,\mathcal{A}$ (resp., $\mathsf{Proj}\,\mathcal{A}$) be the full subcategory of $\mathcal{A}$ consisting of injective (resp., projective) objects.

**Lemma 10.8.** *For* $X^\cdot \in \mathsf{K}(\mathcal{A})$ *and* $I^\cdot \in \mathsf{K}^+(\mathsf{Inj}\,\mathcal{A})$ *(resp.,* $P^\cdot \in \mathsf{K}^-(\mathsf{Proj}\,\mathcal{A})$*), if* $X^\cdot$ *is acyclic, then we have*

$$\operatorname{Hom}_{\mathsf{K}(\mathcal{A})}(X^\cdot, I^\cdot) = 0.$$
$$(\textit{resp., } \operatorname{Hom}_{\mathsf{K}(\mathcal{A})}(P^\cdot, X^\cdot) = 0)$$

**Corollary 10.9.** *The following hold.*

1. *If $\mathcal{A}$ has enough injectives, then we have an isomorphism*
$$\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X^{\cdot}, I^{\cdot}) \cong \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(X^{\cdot}, I^{\cdot})$$
*for $X^{\cdot} \in \mathsf{K}(\mathcal{A})$, $I^{\cdot} \in \mathsf{K}^{+}(\mathsf{Inj}\,\mathcal{A})$.*
2. *If $\mathcal{A}$ has enough projectives, then we have an isomorphism*
$$\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(P^{\cdot}, Y^{\cdot}) \cong \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(P^{\cdot}, Y^{\cdot})$$
*for $P^{\cdot} \in \mathsf{K}^{-}(\mathsf{Proj}\,\mathcal{A})$, $Y^{\cdot} \in \mathsf{K}(\mathcal{A})$.*

*Proof.* By Lemma 10.8 and Proposition 9.8, and their dual. □

**Lemma 10.10.** *The following hold.*

1. *Let $\mathcal{L}$ be a collection of objects of $\mathcal{A}$ such that every object $X \in \mathcal{A}$ is a image of an epimorphism from some object of $\mathcal{L}$. Then for any $X^{\cdot} \in \mathsf{K}^{-}(\mathcal{A})$, there exists $P^{\cdot} \in \mathsf{K}^{-}(\mathcal{L})$ and a morphism $f : P^{\cdot} \to X^{\cdot}$ in $\mathsf{K}(\mathcal{A})$ such that $f$ is a quasi-isomorphism.*
2. *Let $\mathcal{A}'$ be a thick abelian full subcategory of $\mathcal{A}$ such that every object $X \in \mathcal{A}'$ is a image of an epimorphism from some object of $(\mathsf{Proj}\,\mathcal{A}) \cap \mathcal{A}'$. Then for any $X^{\cdot} \in \mathsf{K}^{-}_{\mathcal{A}'}(\mathcal{A})$, there exists $P^{\cdot} \in \mathsf{K}^{-}((\mathsf{Proj}\,\mathcal{A}) \cap \mathcal{A}')$ and a morphism $f : P^{\cdot} \to X^{\cdot}$ in $\mathsf{K}(\mathcal{A})$ such that $f$ is a quasi-isomorphism.*

*Proof.* 1. Given an object $X \in \mathsf{K}^{-}(\mathcal{A})$, we may assume that $X^i = O$ for all $i > 0$. By the backward induction on $n$, we construct a complex $P^{\cdot} \in \mathsf{K}^{-}(\mathsf{Proj}\,\mathcal{A})$. as follows. Let $Z'^n$ and $Z^n$ be $\mathrm{Z}^n(P^{\cdot})$ and $\mathrm{Z}^n(X^{\cdot})$, respectively. Assume we have a commutative diagram

$$\begin{array}{ccc} Z'^n & \rightarrowtail & P^n \\ \downarrow & & \downarrow \\ Z^n & \rightarrowtail & X^n \end{array}$$

We take a pull back $M^n$ of $X^{n-1} \to Z^n \leftarrow Z'^n$, and take an epimorphism from $P^{n-1} \to M^n$. Then by Proposition 2.19, $\mathrm{H}^n(P^{\cdot}) \cong \mathrm{H}^n(X^{\cdot})$ and the induced morphism $\mathrm{Z}^n(P^{\cdot}) \to \mathrm{Z}^n(X^{\cdot})$ is epic.

$$\begin{array}{ccccccccc} Z'^{n-1} & \rightarrowtail & P^{n-1} & \twoheadrightarrow & M^{n-1} & \longrightarrow & Z'^n & \rightarrowtail & P^n \\ \downarrow & & \downarrow & & \downarrow & PB & \downarrow & & \downarrow \\ Z^{n-1} & \rightarrowtail & X^{n-1} & = & X^{n-1} & \longrightarrow & Z^n & \rightarrowtail & X^n \end{array}$$

2. Given an object $X \in \mathsf{K}^{-}_{\mathcal{A}'}(\mathcal{A})$, we may assume that $X^i = O$ for all $i > 0$. By the backward induction on $n$, we construct a complex $P^{\cdot} \in \mathsf{K}^{-}((\mathsf{Proj}\,\mathcal{A}) \cap \mathcal{A}')$. as follows. Let $B'^n$ and $B^n$ (resp., $C'^n$ and $C^n$) be $\mathrm{Z}^n(P^{\cdot})$ and $\mathrm{Z}^n(X^{\cdot})$ (resp., $\mathrm{C}^n(P^{\cdot})$ and $\mathrm{C}^n(X^{\cdot})$, respectively. Assume we have a commutative diagram

$$\begin{array}{ccc} P^n & \twoheadrightarrow & C'^n \\ \downarrow & & \downarrow \\ X^n & \twoheadrightarrow & C^n \end{array}$$

where $P^n, C'^n \in \mathcal{A}'$. Then $B'^n \in \mathcal{A}'$. We take a pull back $C'^{n-1}$ of $C^{n-1} \to B^n \leftarrow B'^n$. Then by Proposition 2.19, $\mathrm{H}^{n-1}(P^{\cdot}) \cong \mathrm{H}^{n-1}(X^{\cdot})$. Since $\mathcal{A}'$ is closed under extensions, $C'^{n-1} \in \mathcal{A}'$. Therefore we can take an epimorphism from $P^{n-1} \to$

${C'}^{n-2}$, with $P^{n-1} \in (\mathsf{Proj}\,\mathcal{A}) \cap \mathcal{A}'$. Since $P^{n-1}$ is projective, we have a morphism $P^{n-1} \to X^{n-1}$, and we have a commutative diagram

$$\begin{array}{ccccccccc}
P^{n-1} & \twoheadrightarrow & {C'}^{n-1} & \twoheadrightarrow & {B'}^{n-1} & \rightarrowtail & P^{n} & \twoheadrightarrow & {C'}^{n} \\
\downarrow & & \downarrow & PB & \downarrow & & \downarrow & & \downarrow \\
X^{n-1} & \twoheadrightarrow & C^{n-1} & \twoheadrightarrow & B^{n-1} & \rightarrowtail & X^{n} & \twoheadrightarrow & C^{n}
\end{array}$$

□

**Proposition 10.11.** *The following hold.*

1. *If $\mathcal{A}$ has enough projectives, then*

$$\mathsf{K}^{-}(\mathsf{Proj}\,\mathcal{A}) \overset{t}{\cong} \mathsf{D}^{-}(\mathcal{A}).$$

2. *If $\mathcal{A}$ has enough injectives, then*

$$\mathsf{K}^{+}(\mathsf{Inj}\,\mathcal{A}) \overset{t}{\cong} \mathsf{D}^{+}(\mathcal{A}).$$

3. *Let $\mathcal{A}'$ be a thick abelian full subcategories of $\mathcal{A}$ such that $\mathcal{A}'$ has enough $\mathcal{A}$-projectives in $\mathcal{A}'$. Then we have*

$$\mathsf{D}^{*}(\mathcal{A}') \overset{t}{\cong} \mathsf{D}^{*}_{\mathcal{A}'}(\mathcal{A})$$

*where $* = -, b$.*

4. *Let $\mathcal{A}'$ be a thick abelian full subcategories of $\mathcal{A}$ such that $\mathcal{A}'$ has enough $\mathcal{A}$-injectives in $\mathcal{A}'$. Then we have*

$$\mathsf{D}^{*}(\mathcal{A}') \overset{t}{\cong} \mathsf{D}^{*}_{\mathcal{A}'}(\mathcal{A})$$

*where $* = +, b$.*

*Proof.* 1. By Lemmas 10.8, 10.10, $(\mathsf{K}^{-}(\mathsf{Proj}\,\mathcal{A}), \mathsf{K}^{-,\phi}(\mathcal{A}))$ is a stable $t$-structure in $\mathsf{K}^{-}(\mathcal{A})$. According to Proposition 9.16, Remark 9.17, we get the statement.

2. Similarly.

3. Since we have the canonical full embedding $\mathsf{K}^{-}(\mathcal{A}') \to \mathsf{K}^{-}(\mathcal{A})$, it suffices to check the condition of Proposition 7.16. Let $X^{\cdot} \in \mathsf{K}^{-}(\mathcal{A}')$, $Y^{\cdot} \in \mathsf{K}^{-}(\mathcal{A})$, and $Y^{\cdot} \to X^{\cdot}$ a quasi-isomorphism in $\mathsf{K}^{-}(\mathcal{A})$. Since all homologies of $Y^{\cdot}$ are in $\mathcal{A}'$, by Lemma 10.10, we have $X'^{\cdot} \to Y^{\cdot}$ is a quasi-isomorphism, with $X'^{\cdot} \in \mathsf{D}^{-}(\mathcal{A}')$.

4. Similarly. □

**Definition 10.12.** In the case of $\mathcal{A}$ having enough projectives (resp., injectives), we denote by $\mathsf{K}^{-,\mathrm{b}}(\mathsf{Proj}\,\mathcal{A})$ (resp., $\mathsf{K}^{+,\mathrm{b}}(\mathsf{Inj}\,\mathcal{A})$) the triangulated full subcategory of $\mathsf{K}^{-}(\mathsf{Proj}\,\mathcal{A})$ (resp., $\mathsf{K}^{+}(\mathsf{Inj}\,\mathcal{A})$) consisting of complexes of which homologies are bounded.

**Corollary 10.13.** *The following hold.*

1. *If $\mathcal{A}$ has enough projectives, then*

$$\mathsf{K}^{-,\mathrm{b}}(\mathsf{Proj}\,\mathcal{A}) \overset{t}{\cong} \mathsf{D}^{\mathrm{b}}(\mathcal{A}).$$

2. *If $\mathcal{A}$ has enough injectives, then*

$$\mathsf{K}^{+,\mathrm{b}}(\mathsf{Inj}\,\mathcal{A}) \overset{t}{\cong} \mathsf{D}^{\mathrm{b}}(\mathcal{A}).$$

**Example 10.14.** For a coherent ring $A$, let $\mathsf{mod}\, A$ be the full subcategory of $\mathsf{Mod}\, A$ consisting of right coherent $A$-modules. Then $\mathsf{mod}\, A$ is an thick abelian full subcategory of $\mathsf{Mod}\, A$. Therefore, we have

$$\mathsf{D}^*(\mathsf{mod}\, A) \overset{t}{\cong} \mathsf{D}^*_{\mathsf{mod}\, A}(\mathsf{Mod}\, A)$$

where $* = -, b$. We often write $\mathsf{D}^*_c(\mathsf{Mod}\, A)$ for $\mathsf{D}^*_{\mathsf{mod}\, A}(\mathsf{Mod}\, A)$.

**Definition 10.15** (Yoneda Ext)**.** For $X, Y \in \mathcal{A}$ and $n \in \mathbb{N}$, let $\mathrm{Exact}^n_{\mathcal{A}}(X, Y)$ be the set of exact sequences in $\mathcal{A}$ of the form

$$\Sigma : O \to Y \to X_{n-1} \to \dots \to X_0 \to X \to O.$$

For $\Sigma_1, \Sigma_2 \in \mathrm{Exact}^n_{\mathcal{A}}(X, Y)$, we write $\Sigma_1 \to \Sigma_2$ if we have a commutative diagram

$$\begin{array}{ccccccccccccc}
\Sigma_1 : O & \longrightarrow & Y & \longrightarrow & X_{n-1} & \longrightarrow & \dots & \longrightarrow & X_0 & \longrightarrow & X & \longrightarrow & O \\
\downarrow & & \| & & \downarrow & & \dots & & \downarrow & & \| & & \\
\Sigma_2 : O & \longrightarrow & Y & \longrightarrow & X'_{n-1} & \longrightarrow & \dots & \longrightarrow & X'_0 & \longrightarrow & X & \longrightarrow & O.
\end{array}$$

And, we define $\Sigma_1 \sim \Sigma_m$ if there are $\Sigma_i$ $(2 \le i \le m-1)$ such that $\Sigma_i \rightleftarrows \Sigma_{i+1}$ $(1 \le i \le m-1)$, where $\rightleftarrows$ means $\to$ or $\leftarrow$. Then $\sim$ is a equivalent relation on $\mathrm{Exact}^n_{\mathcal{A}}(X, Y)$. We denote $\mathrm{Exact}^n_{\mathcal{A}}(X, Y)/\sim$ by $\mathrm{Ext}^n_{\mathcal{A}}(X, Y)$.

**Proposition 10.16.** *For $X, Y \in \mathcal{A}$ and $n \in \mathbb{N}$, We have a bifunctorial isomorphism*

$$\mathrm{Ext}^n_{\mathcal{A}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(X, T^n Y).$$

*Proof.* Let $\Sigma \in \mathrm{Exact}^n_{\mathcal{A}}(X, Y)$ has the form

$$\Sigma : O \to Y \to X_{n-1} \to \dots \to X_0 \to X \to O.$$

Then we have a commutative diagram

$$\begin{array}{cccccccccccccc}
Y[n-1] : & O & \longrightarrow & Y & \longrightarrow & O & & & & & & & & \\
\downarrow & \downarrow & & \downarrow & & \downarrow & & & & & & & & \\
M^{\bullet} : & O & \longrightarrow & X_{n-1} & \longrightarrow & X_{n-2} & \longrightarrow & \dots & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & O \\
\downarrow & & & & & & & & & \downarrow & & \downarrow & & \downarrow \\
X : & & & & & & & & & O & \longrightarrow & X & \longrightarrow & O
\end{array}$$

Therefore, we have a triangle $Y[n-1] \to M^{\bullet} \to X \xrightarrow{\phi(\Sigma)} Y[n]$. It is easy to see that $\phi_{X,Y} : \mathrm{Ext}^n_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(X, Y[n])$ is a bifunctorial isomorphism (left to the reader). $\square$

**Remark 10.17.** Assume $\mathcal{A}$ has enough injectives. For $X, Y \in \mathcal{A}$, let

$$O \to Y \to I^0 \to I^1 \to \dots$$

be an injective resolution, that is, $Y \to I^{\bullet}$ is a quasi-isomorphism. Then by Corollary 10.9, it is easy to see that

$$\begin{aligned}
\mathrm{Ext}^n_{\mathcal{A}}(X, Y) &\cong \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(X, T^n Y) \\
&\cong \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(X, T^n I^{\bullet}) \\
&\cong \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X, T^n I^{\bullet}) \\
&\cong \mathrm{H}^n(\mathrm{Hom}_{\mathcal{A}}(X, I^{\bullet})).
\end{aligned}$$

The last term is $\mathrm{Ext}^n_{\mathcal{A}}(X, Y)$ in the sense of standard homological algebra.

**Definition 10.18.** Let $\mathcal{A}$ be an abelian category with enough injectives. A complex $X^{\cdot} \in \mathsf{K}^*(\mathcal{A})$ is said to have *finite injective dimension* if there is $n \in \mathbb{Z}$ such that $\mathrm{Hom}_{\mathsf{D}^*(\mathcal{A})}(M, X^{\cdot}[i]) = 0$ for all $M \in \mathcal{A}$ and $i > n$, where $* =$ nothing, $+, -, b$.

We denote by $\mathsf{K}^*(\mathcal{A})_{\mathrm{fid}}$ the full subcategory of $\mathsf{K}^*(\mathcal{A})$ consisting of $X^{\cdot} \in \mathsf{K}^*(\mathcal{A})$ which have finite injective dimension.

Moreover, for a thick abelian full subcategory $\mathcal{A}'$ of $\mathcal{A}$, we denote by $\mathsf{K}^*_{\mathcal{A}'}(\mathcal{A})_{\mathrm{fid}}$ the full subcategory of $\mathsf{K}^*_{\mathcal{A}'}(\mathcal{A})$ consisting of $X^{\cdot} \in \mathsf{K}^*_{\mathcal{A}'}(\mathcal{A})$ which have finite injective dimension.

**Proposition 10.19.** *Let $\mathcal{A}$ be an abelian category with enough injectives. Then $\mathsf{K}^*(\mathcal{A})_{\mathrm{fid}}$ and $\mathsf{K}^*_{\mathcal{A}'}(\mathcal{A})_{\mathrm{fid}}$ are quotientizing subcategories of $\mathsf{K}^*(\mathcal{A})$.*

*Proof.* For $u : X^{\cdot} \to Y^{\cdot}$ in $\mathsf{K}^*(\mathcal{A})_{\mathrm{fid}}$, let $X^{\cdot} \xrightarrow{u} Y^{\cdot} \to \mathrm{M}^{\cdot}(u) \to X^{\cdot}[1]$ be a triangle. For $M \in \mathcal{A}$, by applying $\mathrm{Hom}_{\mathsf{K}^*(\mathcal{A})}(M, -)$ to the triangle, we have $\mathrm{M}^{\cdot}(u) \in \mathsf{K}^*(\mathcal{A})_{\mathrm{fid}}$. Therefore, $\mathsf{K}^*(\mathcal{A})_{\mathrm{fid}}$ is a triangulated full subcategory of $\mathsf{K}^*(\mathcal{A})$. By Proposition 9.2, $\mathsf{K}^{*,\phi}(\mathcal{A})_{\mathrm{fid}} = \mathsf{K}^{\phi}(\mathcal{A}) \cap \mathsf{K}^*(\mathcal{A})_{\mathrm{fid}}$ is an épaisse subcategory of $\mathsf{K}^*(\mathcal{A})$. According to Proposition 7.16, $\mathsf{K}^*(\mathcal{A})_{\mathrm{fid}}$ is a quotientizing subcategory of $\mathsf{K}^*(\mathcal{A})$. In the case of $\mathsf{K}^*_{\mathcal{A}'}(\mathcal{A})_{\mathrm{fid}}$, similarly. □

**Definition 10.20.** For $* =$ nothing, $+, -, b$, $\mathsf{D}^*(\mathcal{A})_{\mathrm{fid}} = \mathsf{K}^*(\mathcal{A})_{\mathrm{fid}}/\mathsf{K}^{*,\phi}(\mathcal{A})_{\mathrm{fid}}$ and $\mathsf{D}^*_{\mathcal{A}'}(\mathcal{A})_{\mathrm{fid}} = \mathsf{K}^*_{\mathcal{A}'}(\mathcal{A})_{\mathrm{fid}}/\mathsf{K}^{*,\phi}(\mathcal{A})_{\mathrm{fid}}$.

**Proposition 10.21.** *Let $\mathcal{A}$ be an abelian category with enough injectives. Then the following are equivalent for $X^{\cdot} \in \mathsf{K}^+(\mathcal{A})$.*

1. *For any integer $n_1 \in \mathbb{Z}$ there is $n_2 \in \mathbb{Z}$ such that $\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(Y^{\cdot}, X^{\cdot}[i]) = 0$ for all $i > n_2$ and all complexes $Y^{\cdot} \in K^+(\mathcal{A})$ with $\mathrm{H}^j(Y^{\cdot}) = O$ for $j < n_1$.*
2. *$X^{\cdot} \in \mathsf{K}^+(\mathcal{A})_{\mathrm{fid}}$.*
3. *There exists $I^{\cdot} \in \mathsf{K}^{\mathrm{b}}(\mathsf{Inj}\,\mathcal{A})$ such that $X^{\cdot} \cong I^{\cdot}$ in $\mathsf{D}^+(\mathcal{A})$.*

*Proof.* $1 \Rightarrow 2$. It is trivial.

$2 \Rightarrow 3$. Let $n$ be an integer such that $\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(M, X^{\cdot}[i]) = 0$ for all $i > n$. We take $I^{\cdot} \in \mathsf{K}^+(\mathsf{Inj}\,\mathcal{A})$ which has a quasi-isomorphism $X^{\cdot} \to I^{\cdot}$ in $\mathsf{K}^+(\mathcal{A})$. For $i > n$, we have isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(\mathrm{Z}^i(I^{\cdot})[-i], X^{\cdot}) &\cong \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(\mathrm{Z}^i(I^{\cdot}), I^{\cdot}[i]) \\ &= 0.\end{aligned}$$

This means that the canonical morphisms $I^{i-1} \to \mathrm{Z}^i(I^{\cdot})$ is split epic, and $\mathrm{B}^i(I^{\cdot}) = \mathrm{Z}^i(I^{\cdot})$. Then $\sigma_{\leq n} I^{\cdot} \to I^{\cdot}$ is an isomorphism in $\mathsf{K}(\mathcal{A})$ and $\sigma_{\leq n} I^{\cdot} \in \mathsf{K}^{\mathrm{b}}(\mathsf{Inj}\,\mathcal{A})$.

$3 \Rightarrow 1$. By Corollary 10.9, it is easy. □

**Corollary 10.22.** *Let $\mathcal{A}$ be an abelian category with enough injectives. Then we have a triangle equivalent*

$$\mathsf{K}^{\mathrm{b}}(\mathsf{Inj}\,\mathcal{A}) \overset{t}{\cong} \mathsf{D}^+(\mathcal{A})_{\mathrm{fid}}.$$

**Proposition 10.23.** *Let $\mathcal{B}$ be an additive full subcategory of an abelian category $\mathcal{A}$ which is closed under direct summands, and $\tilde{\mathsf{K}}^{\mathrm{b}}(\mathcal{B})$ the triangulated full subcategory of $\mathsf{K}^-(\mathcal{B})$ consisting of objects which are isomorphic to an object of $\mathsf{K}^{\mathrm{b}}(\mathcal{B})$ in $\mathsf{K}^-(\mathcal{B})$. Then $\tilde{\mathsf{K}}^{\mathrm{b}}(\mathcal{B})$ is an épaisse subcategory of $\mathsf{K}^-(\mathcal{B})$.*

*Proof.* Let $X^{\cdot} \in \mathsf{K}^{\mathrm{b}}(\mathcal{B})$, and $Y^{\cdot}$ is direct summand of $X^{\cdot}$ in $\mathsf{K}^{-}(\mathcal{B})$. Then $Y^{\cdot}$ is a direct summand of $X^{\cdot}\oplus I^{\cdot}(Y^{\cdot})$ in $\mathsf{C}^{-}(\mathcal{B})$. Then we have a split exact sequence in $\mathsf{C}^{-}(\mathcal{B})$

$$O \to Y^{\cdot} \to X^{\cdot}\oplus I^{\cdot}(Y^{\cdot}) \to Z^{\cdot} \to O.$$

Since $X^{\cdot} \in \mathsf{K}^{\mathrm{b}}(\mathcal{B})$, there is $n \in \mathbb{Z}$ such that we have a split exact sequence in $\mathsf{C}^{-}(\mathcal{B})$

$$O \to \tau_{\leq n}Y^{\cdot} \to \tau_{\leq n}I^{\cdot}(Y^{\cdot}) \to \tau_{\leq n}Z^{\cdot} \to O.$$

Since $\mathrm{H}^i(\tau_{\leq n}I^{\cdot}(Y^{\cdot})) = O$ for $i \neq n$ and $\mathrm{B}^i(\tau_{\leq n}I^{\cdot}(Y^{\cdot})) \in \mathcal{B}$, $\mathrm{H}^i(\tau_{\leq n}Y^{\cdot}) = O$ for $i \neq n$ and $\mathrm{B}^i(\tau_{\leq n}Y^{\cdot}) \in \mathcal{B}$. Hence $Y^{\cdot} \cong \sigma_{>n-2}Y^{\cdot}$ in $\mathsf{K}^{-}(\mathcal{B})$ with $\sigma_{>n-2}Y^{\cdot} \in \mathsf{K}^{\mathrm{b}}(\mathcal{B})$. □

## 11. Homotopy Limits

Throughout this section $\mathcal{C}$ is a triangulated category with arbitrary coproducts.

**Definition 11.1.** A triangulated full subcategory $\mathcal{L}$ of $\mathcal{C}$ is called *localizing* if

(L1) Every direct summand of an object in $\mathcal{L}$ is in $\mathcal{L}$.

(L2) Every coproduct of objects in $\mathcal{L}$ is in $\mathcal{L}$.

**Lemma 11.2.** *Let $\mathcal{L}$ of $\mathcal{C}$ be a localizing subcategory, then $\mathcal{C}/\mathcal{L}$ has arbitrary coproducts, and the quotient $\mathcal{C} \to \mathcal{C}/\mathcal{L}$ preserves coproducts.*

*Proof.* Let $\{X_I\}_{i\in I}$ be a collection of objects of $\mathcal{C}$. It suffices to show

1. Any collection of morphisms $X_i \xrightarrow{f_i} Y$ in $\mathcal{C}/\mathcal{L}$ can be lifted to a morphism $\coprod_i X_i \xrightarrow{f} Y$ in $\mathcal{C}/\mathcal{L}$.
2. a morphism $\coprod_i X_i \xrightarrow{f} Y$ in $\mathcal{C}/\mathcal{L}$ such that all $X_i \xrightarrow{q_i} \coprod_i X_i \xrightarrow{f} Y = 0$ in $\mathcal{C}/\mathcal{L}$, then $f = 0$.

1. A morphism $X_i \xrightarrow{f_i} Y$ in $\mathcal{C}/\mathcal{L}$ is a diagram in $\mathcal{C}$

$$\begin{array}{ccc} X_i^{'} & & \\ {\scriptstyle s_i}\downarrow & \searrow^{f_i^{'}} & \\ X_i & & Y \end{array}$$

where $X_i \to X_i \to X_i^{''} \to TX_i^{'}$ is a triangle, with $X''_i \in \mathcal{L}$. Thus we get a diagram

$$\begin{array}{ccc} \coprod_i X_i^{'} & & \\ {\scriptstyle \coprod s_i}\downarrow & \searrow^{\sum f_i^{'}} & \\ \coprod_i X_i & & Y. \end{array}$$

Since $\coprod_i X_i \to \coprod_i X_i \to \coprod_i X_i^{''} \to T\coprod_i X_i^{'}$ is a triangle, we have a morphism $\coprod_i X_i \xrightarrow{f} Y$ in $\mathcal{C}/\mathcal{L}$.

2. Given a morphism $\coprod_i X_i \xrightarrow{f} Y$ in $\mathcal{C}/\mathcal{L}$, it corresponds to a diagram

$$\begin{array}{ccc} \coprod_i X_i & & Y \\ & \searrow^{f'} & \downarrow{\scriptstyle t} \\ & & Y \end{array}$$

with $t \in \Phi(\mathcal{L})$. If the composite $X_i \to \coprod_i X_i \to Y = 0$ in $\mathcal{C}/\mathcal{L}$, then we have a diagram in $\mathcal{C}$

$$\begin{array}{ccccc} X_i & \xrightarrow{q_i} & \coprod_i X_i & & Y \\ & & & \searrow^{f'} & \downarrow t \\ & & & & Y' \end{array}$$

which corresponds to 0 in $\mathcal{C}/\mathcal{L}$. Then by Proposition 7.11 and Lemma 9.4, every $X_i \xrightarrow{q_i} \coprod_i X_i \xrightarrow{f'} Y'$ factors through $Z_i \in \mathcal{L}$. Thus $f$ factorizes as

$$\begin{array}{ccccc} \coprod_i X_i & \longrightarrow & \coprod_i Z_i & & Y \\ & & & \searrow & \downarrow t \\ & & & & Y'. \end{array}$$

Since $\mathcal{L}$ is localizing, $\coprod_i Z_i \in \mathcal{L}$ and $f = 0$. □

**Corollary 11.3.** *Let $\mathcal{A}$ be an abelian category satisfying the condition Ab4. Then $\mathsf{D}(\mathcal{A})$ has arbitrary coproducts.*

*Proof.* According to Corollary 6.15, $\mathsf{K}(\mathcal{A})$ has arbitrary coproducts. For a collection of quasi-isomorphisms $X_i^{\cdot} \xrightarrow{f_i} Y_i^{\cdot}$ $(i \in I)$, $\mathrm{H}^n(\coprod_i f_i) \cong \coprod_i \mathrm{H}^n(f_i)$ is isomorphic for all $n \in \mathbb{Z}$. Thus $\mathsf{K}^{\phi}(\mathcal{A})$ is localizing, and Lemma 11.2 can be applied. □

**Definition 11.4.** For a sequence $\{X_i \to X_{i+1}\}_{i\in\mathbb{N}}$ (resp., $\{X_{i+1} \to X_i\}_{i\in\mathbb{N}}$) of morphisms in $\mathcal{C}$, the homotopy colimit (resp., limit) of the sequence is the third (resp., second) term of the triangle

$$\coprod_i X_i \xrightarrow{1-\text{ shift}} \coprod_i X_i \to \underset{\longrightarrow}{\text{hlim}}\, X_i \to T\coprod_i X_i$$
$$(\text{resp., } T^{-1}\prod_i X_i \to \underset{\longleftarrow}{\text{hlim}}\, X_i \to \prod_i X_i \xrightarrow{1-\text{ shift}} \prod_i X_i)$$

where the above shift morphism is the coproduct (resp., product) of $X_i \xrightarrow{f_i} X_{i+1}$ (resp., $X_{i+1} \xrightarrow{f_i} X_i$) $(i \in \mathbb{N})$.

**Exercise 11.5.** In the category $\mathfrak{Ab}$, prove the following.

1. For a sequence of morphisms $\{X_{i+1} \xrightarrow{f_i} X_i\}_{i\in\mathbb{N}}$, if there is $n \in \mathbb{N}$ such that $f_i$ are epimorphisms for all $i \geq n$, then $\prod_i X_i \xrightarrow{1-\text{ shift}} \prod_i X_i$ is epic.
2. For a sequence of morphisms $\{X_i \xrightarrow{f_i} X_{i+1}\}_{i\in\mathbb{N}}$, $\coprod_i X_i \xrightarrow{1-\text{ shift}} \coprod_i X_i$ is monic.

**Lemma 11.6.** *The following hold.*

1. *Assume $\mathcal{A}$ satisfies the condition Ab3. For a sequence of morphisms $\{X_i \xrightarrow{f_i} X_{i+1}\}_{i\in\mathbb{N}}$, if there is $n \in \mathbb{N}$ such that $f_i$ are split monomorphisms for all $i \geq n$, then we have a split exact sequence*

$$O \to \coprod_i X_i \xrightarrow{1-\ shift} \coprod_i X_i \to \underset{\longrightarrow}{\lim}\, X_i \to O.$$

2. *Assume $\mathcal{A}$ satisfies the condition Ab3*. For a sequence of morphisms $\{X_{i+1} \xrightarrow{f_i} X_i\}_{i\in\mathbb{N}}$, if there is $n \in \mathbb{N}$ such that $f_i$ are split epimorphisms for all $i \geq n$, then we have a split exact sequence*

$$O \to \varprojlim X_i \to \prod_i X_i \xrightarrow{1-\ shift} \prod_i X_i \to O.$$

*Proof.* 1, For any $M \in \mathcal{A}$, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(\coprod_i X_i, M) & \xrightarrow{1-\text{ shift}} & \mathrm{Hom}_{\mathcal{A}}(\coprod_i X_i, M) \\ \downarrow \wr & & \downarrow \wr \\ \prod_i \mathrm{Hom}_{\mathcal{A}}(X_i, M) & \xrightarrow{1-\text{ shift}} & \prod_i \mathrm{Hom}_{\mathcal{A}}(X_i, M). \end{array}$$

By Exercise 11.5, the bottom horizontal morphism is epic.
2. Similarly. □

**Proposition 11.7.** *The following hold.*

1. *Assume $\mathcal{A}$ satisfies the condition Ab3. and $X_i^{\cdot} \to X_{i+1}^{\cdot}$ a sequence of complexes in $\mathsf{C}(\mathcal{A})$ satisfying that for each $j \in \mathbb{Z}$ there is $n \in \mathbb{N}$ such that $X_i^j \to X_{i+1}^j$ are split monomorphisms for all $i \geq n$. Then we have an exact sequence in $\mathsf{C}(\mathcal{A})$*

$$O \to \coprod_i X_i^{\cdot} \xrightarrow{1-\ shift} \coprod_i X_i^{\cdot} \to \varinjlim X_i^{\cdot} \to O$$

*which belongs to $\mathsf{S}_{\mathsf{C}(\mathcal{A})}$. In particular, $\varinjlim X_i^{\cdot} \cong \underrightarrow{\mathrm{hlim}}\ X_i^{\cdot}$ in $\mathsf{K}(\mathcal{A})$.*

2. *Assume $\mathcal{A}$ satisfies the condition Ab3*. and $X_{i+1}^{\cdot} \to X_i^{\cdot}$ a sequence of complexes in $\mathsf{C}(\mathcal{A})$ satisfying that for each $j \in \mathbb{Z}$ there is $n \in \mathbb{N}$ such that $X_i^j \to X_{i+1}^j$ are split epimorphisms for all $i \geq n$. Then we have an exact sequence in $\mathsf{C}(\mathcal{A})$*

$$O \to \varprojlim X_i^{\cdot} \to \prod_i X_i^{\cdot} \xrightarrow{1-\ shift} \prod_i X_i^{\cdot} \to O$$

*which belongs to $\mathsf{S}_{\mathsf{C}(\mathcal{A})}$. In particular, $\varprojlim X_i^{\cdot} \cong \underleftarrow{\mathrm{hlim}}\ X_i^{\cdot}$ in $\mathsf{K}(\mathcal{A})$.*

*Proof.* 1. By Lemma 11.6, for any $j$ we have a split exact sequence

$$O \to \coprod_i X_i^j \xrightarrow{1-\text{ shift}} \coprod_i X_i^j \to \varinjlim X_i^j \to O.$$

Then we have an exact sequence in $\mathsf{C}(\mathcal{A})$

$$O \to \coprod_i X_i^{\cdot} \xrightarrow{1-\text{ shift}} \coprod_i X_i^{\cdot} \to \varinjlim X_i^{\cdot} \to O$$

which belongs to $\mathsf{S}_{\mathsf{C}(\mathcal{A})}$. The last assertion follows by Proposition 6.12.
2. Similarly. □

**Remark 11.8.** The above $\varinjlim X_i^{\cdot}$ and $\varprojlim X_i^{\cdot}$ are the filtered colimit and the filtered limit in $\mathsf{C}(\mathcal{A})$, but are not the filtered colimit and the filtered limit in $\mathsf{K}(\mathcal{A})$ (see Lemma 16.17).

**Remark 11.9.** 1. If $\mathcal{A}$ satisfies the condition Ab5, then for a sequence $\{X_i^{\cdot} \to X_{i+1}^{\cdot}\}_{i\in\mathbb{N}}$ of morphisms in $\mathsf{D}(\mathcal{A})$, we have exact sequences

$$O \to \coprod_i \mathrm{H}^n(X_i^{\cdot}) \to \coprod_i \mathrm{H}^n(X_i^{\cdot}) \to \mathrm{H}^n(\underrightarrow{\mathrm{hlim}}\ X_i^{\cdot}) \to O$$

for all $n \in \mathbb{N}$.

2. If $\mathcal{A}$ satisfies the condition Ab5 and $\{X_i^\cdot \to X_{i+1}^\cdot\}_{i\in\mathbb{N}}$ a sequence of morphisms in $\mathsf{C}(\mathcal{A})$, then by Proposition 2.25, 3 we have an exact sequence in $\mathsf{C}(\mathcal{A})$

$$O \to \coprod_i X_i^\cdot \to \coprod_i X_i^\cdot \to \varinjlim X_i^\cdot \to O$$

and we have a quasi-isomorphism

$$\underrightarrow{\text{hlim}}\; X_i^\cdot \to \varinjlim X_i^\cdot.$$

3. Assume $\mathcal{A}$ satisfies the condition Ab4*, and let $\{X_{i+1}^\cdot \to X_i^\cdot\}_{i\in\mathbb{N}}$ be a sequence of morphisms in $\mathsf{D}(\mathcal{A})$ satisfying that for any $n \in \mathbb{Z}$ there is $k \in \mathbb{N}$ such that $\mathrm{H}^n(X_{i+1}^\cdot) \cong \mathrm{H}^n(X_i^\cdot)$ for all $i > k$. Then we have exact sequences

$$O \to \mathrm{H}^n(\underleftarrow{\text{hlim}}\; X_i^\cdot) \to \prod_i \mathrm{H}^n(X_i^\cdot) \to \prod_i \mathrm{H}^n(X_i^\cdot) \to O$$

for all $n \in \mathbb{N}$.

**Proposition 11.10.** *For an abelian category $\mathcal{A}$, the following hold.*

1. *If $\mathcal{A}$ satisfies the condition Ab4 with enough projectives, then every object of $\mathsf{K}(\mathcal{A})$ is quasi-isomorphic to a complex $P^\cdot$ of projectives with* $\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(P^\cdot, \mathsf{K}^\phi(\mathcal{A})) = 0$.
2. *If $\mathcal{A}$ satisfies the condition Ab4* with enough injectives, then every object of $\mathsf{K}(\mathcal{A})$ is quasi-isomorphic to a complex $P^\cdot$ of injectives with* $\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(\mathsf{K}^\phi(\mathcal{A}), I^\cdot) = 0$.

*Proof.* 1. For a complex $X^\cdot \in \mathsf{K}(\mathcal{A})$, we have morphisms of complexes $\sigma_{\leq i}X^\cdot \to \sigma_{\leq i+1}X^\cdot \to X^\cdot$. According to Lemma 10.10, there is $P_i^\cdot \in \mathsf{K}^-(\mathsf{Proj}\,\mathcal{A})$ which has a quasi-isomorphism $P_i^\cdot \to \sigma_{\leq i}X^\cdot$, and we have a commutative diagram in $\mathsf{K}(\mathcal{A})$

$$\begin{array}{ccc} P_i^\cdot & \longrightarrow & P_{i+1}^\cdot \\ \downarrow & & \downarrow \\ \sigma_{\leq i}X^\cdot & \longrightarrow & \sigma_{\leq i+1}X^\cdot \end{array}$$

By Remark 11.9, we have a quasi-isomorphism

$$\underrightarrow{\text{hlim}}\; P_i^\cdot \to \underrightarrow{\text{hlim}}\; \sigma_{\leq i}X^\cdot.$$

By Proposition 11.7, $\underrightarrow{\text{hlim}}\; \sigma_{\leq i}X^\cdot \cong \varinjlim \sigma_{\leq i}X^\cdot = X^\cdot$ in $\mathsf{K}(\mathcal{A})$. By the construction, $\underrightarrow{\text{hlim}}\; P_i^\cdot$ is a complex of projectives. Since we have an exact sequence

$$\prod_i \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(TP_i^\cdot, \mathsf{K}^\phi(\mathcal{A})) \to \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(P^\cdot, \mathsf{K}^\phi(\mathcal{A})) \to \prod_i \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(TP_i^\cdot, \mathsf{K}^\phi(\mathcal{A})),$$

we have $\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(P^\cdot, \mathsf{K}^\phi(\mathcal{A})) = 0$ by Lemma 10.8.

2. Similarly. □

**Definition 11.11.** 1. In the case that $\mathcal{A}$ satisfies the condition Ab4 with enough projectives, we define the triangulated full subcategory $\mathsf{K}^{\mathrm{s}}(\mathsf{Proj}\,\mathcal{A})$ of $\mathsf{K}(\mathcal{A})$ consisting of complexes $P^\cdot$ of projectives such that $\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(P^\cdot, \mathsf{K}^\phi(\mathcal{A})) = 0$. Then $(\mathsf{K}^{\mathrm{s}}(\mathsf{Proj}\,\mathcal{A}), \mathsf{K}^\phi(\mathcal{A}))$ is a stable $t$-structure in $\mathsf{K}(\mathcal{A})$.

2. In the case that $\mathcal{A}$ satisfies the condition Ab4* with enough injectives, we define the triangulated full subcategory $\mathsf{K}^{\mathrm{s}}(\mathsf{Inj}\,\mathcal{A})$ of $\mathsf{K}(\mathcal{A})$ consisting of complexes $I^\cdot$ of injectives such that $\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(\mathsf{K}^\phi(\mathcal{A}), I^\cdot) = 0$. Then $(\mathsf{K}^\phi(\mathcal{A}), \mathsf{K}^{\mathrm{s}}(\mathsf{Inj}\,\mathcal{A}))$ is a stable $t$-structure in $\mathsf{K}(\mathcal{A})$.

They are often called $\mathsf{K}$-projective complexes (resp., $\mathsf{K}$-injective complexes).

**Proposition 11.12.** *The following hold.*

1. *If $\mathcal{A}$ satisfies the condition Ab4 with enough projectives, then we have an isomorphism*
$$\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(P^{\cdot}, Y^{\cdot}) \cong \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(P^{\cdot}, Y^{\cdot})$$
*for $P^{\cdot} \in \mathsf{K}^{\mathrm{s}}(\mathsf{Proj}\,\mathcal{A})$, $Y^{\cdot} \in \mathsf{K}(\mathcal{A})$.*
2. *If $\mathcal{A}$ satisfies the condition Ab4* with enough injectives, then we have an isomorphism*
$$\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X^{\cdot}, I^{\cdot}) \cong \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(X^{\cdot}, I^{\cdot})$$
*for $X^{\cdot} \in \mathsf{K}(\mathcal{A})$, $I^{\cdot} \in \mathsf{K}^{\mathrm{s}}(\mathsf{Inj}\,\mathcal{A})$.*

**Theorem 11.13.** *The following hold.*

1. *If $\mathcal{A}$ satisfies the condition Ab4 with enough projectives, then we have a triangle equivalence*
$$\mathsf{K}^{\mathrm{s}}(\mathsf{Proj}\,\mathcal{A}) \overset{t}{\cong} \mathsf{D}(\mathcal{A}).$$
2. *If $\mathcal{A}$ satisfies the condition Ab4* with enough injectives, then we have a triangle equivalence*
$$\mathsf{K}^{\mathrm{s}}(\mathsf{Inj}\,\mathcal{A}) \overset{t}{\cong} \mathsf{D}(\mathcal{A}).$$

*Proof.* 1. By Proposition 11.10, $(\mathsf{K}^{\mathrm{s}}(\mathsf{Proj}\,\mathcal{A}), \mathsf{K}^{\phi}(\mathcal{A}))$ is a stable $t$-structure in $\mathsf{K}(\mathcal{A})$. According to Proposition 9.16, Remark 9.17, we get the statement.

2. Similarly. □

**Remark 11.14** (Set-Theoretic Remark 2)**.** Conversely, in the case that $\mathcal{A}$ satisfies the condition Ab4 with enough projectives, we can define $\mathsf{D}^{-}(\mathcal{A}) = \mathsf{K}^{-}(\mathsf{Proj}\,\mathcal{A})$, and $\mathsf{D}(\mathcal{A}) = \mathsf{K}^{\mathrm{s}}(\mathsf{Proj}\,\mathcal{A})$. Then we can bypass Remark 7.7.

Similarly, in the case that $\mathcal{A}$ satisfies the condition Ab4* with enough injectives, we can define $\mathsf{D}^{+}(\mathcal{A}) = \mathsf{K}^{+}(\mathsf{Inj}\,\mathcal{A})$, and $\mathsf{D}(\mathcal{A}) = \mathsf{K}^{\mathrm{s}}(\mathsf{Inj}\,\mathcal{A})$.

**Proposition 11.15.** *The following hold.*

1. *Let $\mathcal{C}$ be a triangulated category with coproducts. For a sequence $\{X_i \xrightarrow{f_i} X_{i+1}\}_{i\in\mathbb{N}}$, if there is $n \in \mathbb{N}$ such that $f_i$ are split monomorphisms for all $i \geq n$, then the structural morphism $\coprod_i X_i \to \underset{\longrightarrow}{\mathrm{hlim}}\, X_i$ is a split epimorphism.*
2. *Let $\mathcal{C}$ be a triangulated category with products. For a sequence $\{X_{i+1} \xrightarrow{f_i} X_i\}_{i\in\mathbb{N}}$, if there is $n \in \mathbb{N}$ such that $f_i$ are split epimorphisms for all $i \geq n$, then the structural morphism $\underset{\longleftarrow}{\mathrm{hlim}}\, X_i \to \prod_i X_i$ is a split monomorphism.*

*Proof.* 1. For any $M \in \mathcal{C}$ and $n \in \mathbb{N}$, we have a commutative diagram
$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(\coprod_i T^n X_i, M) & \xrightarrow{1-\ \mathrm{shift}} & \mathrm{Hom}_{\mathcal{C}}(\coprod_i T^n X_i, M) \\
\downarrow \wr & & \downarrow \wr \\
\prod_i \mathrm{Hom}_{\mathcal{C}}(T^n X_i, M) & \xrightarrow{1-\ \mathrm{shift}} & \prod_i \mathrm{Hom}_{\mathcal{C}}(T^n X_i, M).
\end{array}$$
Then by Exercise 11.5.2, $\mathrm{Hom}_{\mathcal{C}}(\coprod_i T^n X_i, M) \xrightarrow{1-\ \mathrm{shift}} \mathrm{Hom}_{\mathcal{C}}(\coprod_i T^n X_i, M)$ is epic, and then $T^n \coprod_i X_i \xrightarrow{1-\ \mathrm{shift}} T^n \coprod_i X_i$ is split monic. Therefore, $\underset{\longrightarrow}{\mathrm{hlim}}\, X_i \to T\coprod_i X_i = 0$, and hence $\coprod_i X_i \to \underset{\longrightarrow}{\mathrm{hlim}}\, X_i$ is split epic.

2. Similarly. □

**Lemma 11.16.** *Let $\mathcal{C}$ be a triangulated category with coproducts. For a sequence $\{X_i \xrightarrow{f_i} X_{i+1}\}_{i\in\mathbb{N}}$ where $X_i = X$ and $f_i = 1_X$ for all $i$, we have an isomorphism in $\mathcal{C}$*

$$\underrightarrow{\mathrm{hlim}}\ X_i \cong X.$$

*Proof.* Let $\coprod_i X_i \xrightarrow{p} \coprod_i X_i \xrightarrow{q} \underrightarrow{\mathrm{hlim}}\ X_i \xrightarrow{r} \coprod_i TX_i$ be a triangle, and $\alpha = \sum_i (1_X)_i : \coprod_i X_i \to X$. By easy calculation, the following hold.

(a) $\alpha p = 0$.

(b) If a morphism $\phi : \coprod_i X_i \to Y$ satisfies $\phi p = 0$, then there is a unique $f : X \to Y$ such that $\phi = f\alpha$.

By the property of $\underrightarrow{\mathrm{hlim}}$ and the above, there exist $h : X \to \underrightarrow{\mathrm{hlim}}\ X_i$ and $k : \underrightarrow{\mathrm{hlim}}\ X_i \to X$ such that $\alpha = kq$ and $q = h\alpha$. Since $q = hkq$ and $\alpha = kh\alpha$, we have $hk = 1$ and $kh = 1$ by (b) and Proposition 11.15. □

**Proposition 11.17.** *Let $\mathcal{C}$ be a triangulated category with coproducts. Let $e : X \to X$ be a morphism in $\mathcal{C}$ such that $e^2 = e$. Then $e$ splits in $\mathcal{C}$.*

*Proof.* We consider three sequences

(A) $X \xrightarrow{e} X \xrightarrow{e} \dots$.

(B) $X \xrightarrow{1-e} X \xrightarrow{1-e} \dots$.

(C) $X\oplus X \xrightarrow{\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]} X\oplus X \xrightarrow{\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]} \dots$.

Then we have an isomorphism $\alpha = \left[\begin{smallmatrix}e & 1-e\\ 1-e & e\end{smallmatrix}\right] : (A)\oplus(B) \to (C)$ of sequences. Thus $\underrightarrow{\mathrm{hlim}}\ (C) \cong Y \oplus Z$ in $\mathcal{C}$, where $Y = \underrightarrow{\mathrm{hlim}}\ (A)$, $Z = \underrightarrow{\mathrm{hlim}}\ (B)$. For a sequence (C), we have a commutative diagram

$$\begin{array}{ccccc}
\coprod_i (X\oplus X)_i & \xrightarrow{1-\mathrm{shift}} & \coprod_i (X\oplus X)_i & \longrightarrow & \underrightarrow{\mathrm{hlim}}\ (C) \\
\downarrow \wr & & \downarrow \wr & & \\
(\coprod_i X_i)\oplus(\coprod_i X_i) & \xrightarrow{1\oplus(1-\mathrm{shift})} & (\coprod_i X_i)\oplus(\coprod_i X_i). & &
\end{array}$$

By Lemma 11.16, we have $\underrightarrow{\mathrm{hlim}}\ (C) \cong X$ in $\mathcal{C}$. On the other hand, we have a commutative diagram

$$\begin{array}{ccccc}
(\coprod_i X_i)\oplus(\coprod_i X_i) & \xrightarrow{(1-\mathrm{shift})\oplus(1-\mathrm{shift})} & (\coprod_i X_i)\oplus(\coprod_i X_i) & \xrightarrow{\beta} & Y\oplus Z \\
\downarrow \alpha & & \downarrow \alpha & & \downarrow \gamma \\
\coprod_i (X\oplus X)_i & \xrightarrow{1-\mathrm{shift}} & \coprod_i (X\oplus X)_i & \xrightarrow{\sum_i (0\ 1)_i} & X
\end{array}$$

where $\beta = \left[\begin{smallmatrix}a&0\\0&b\end{smallmatrix}\right]$ and $\gamma = [\,\varepsilon\ \eta\,]$ are isomorphisms. According to Proposition 11.15, $(0\ 1)$ and $\beta = \left[\begin{smallmatrix}a&0\\0&b\end{smallmatrix}\right]$ are epic, then $a$ and $b$ are epic. Since $(1-e)\varepsilon b = (1-e)(0\ 1)\left[\begin{smallmatrix}1-e\\e\end{smallmatrix}\right] = 0$, we have $(1-e)\varepsilon = 0$. Similarly, we have $e\eta = 0$. Hence we have a morphism $\delta : X \to Y$ such that $\varepsilon\delta = e$ and $\delta\varepsilon = 1_Y$. □

**Corollary 11.18.** *Let $\mathcal{L}$ be a triangulated full subcategory $\mathcal{L}$ of $\mathcal{C}$. If $\mathcal{L}$ satisfies the condition*

(L2) *every coproduct of objects in $\mathcal{L}$ is in $\mathcal{L}$,* *then $\mathcal{L}$ is a localizing subcategory of $\mathcal{C}$.*

**Corollary 11.19.** *Let $R$ be a commutative complete local ring, $A$ a finite $R$-algebra. Then $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$ is a Krull-Schmidt category.*

*Proof.* For a complex $Y^{\bullet} \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$, we may assume that $\mathrm{H}^m(Y^{\bullet}) \neq 0$, $\mathrm{H}^n(Y^{\bullet}) \neq 0$, and $\mathrm{H}^i(Y^{\bullet}) = 0$ for $i < m$, $n < i$ for some $m < n$. We define $\mathrm{TL}(Y^{\bullet}) = n - m$. For complexes $X^{\bullet}, Y^{\bullet} \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$, by induction on the lexicographic order ($\mathrm{TL}(X^{\bullet})$, $\mathrm{TL}(Y^{\bullet})$), we show that $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)}(X^{\bullet}, Y^{\bullet})$ is a finitely generated $R$-module. Let $Y_1^{\bullet} = \sigma_{\leq n-1}(Y^{\bullet})$ and $Y_2^{\bullet} = \sigma_{>n-1}(Y^{\bullet})$, then we have a triangle in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$

$$Y_1^{\bullet} \to Y^{\bullet} \to Y_2^{\bullet} \to TY_1^{\bullet}.$$

Then we have an exact sequence

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)}(X^{\bullet}, Y_1^{\bullet}) \to \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)}(X^{\bullet}, Y^{\bullet}) \to \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)}(X^{\bullet}, Y_2^{\bullet}).$$

By the assumption, $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)}(X^{\bullet}, Y_1^{\bullet})$ and $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)}(X^{\bullet}, Y_2^{\bullet})$ are finitely generated $R$-modules. Then $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)}(X^{\bullet}, Y^{\bullet})$ is a finitely generated $R$-module. Similarly, for a triangle $\sigma_{\leq n-1}(X^{\bullet}) \to X^{\bullet} \to \sigma_{>n-1}(X^{\bullet}) \to T\sigma_{\leq n-1}(X^{\bullet})$, we have the same result. In particular, $\mathrm{End}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)}(X^{\bullet})$ is a semiperfect ring. For an idempotent $e \in \mathrm{End}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)}(X^{\bullet})$, by Example 10.14, we may consider an idempotent in $\mathsf{D}^{\mathrm{b}}_c(\mathsf{Mod}\, A) \subset \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$. By Proposition 11.17, there exist a complex $Y^{\bullet} \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$ and morphisms $p : X^{\bullet} \to Y^{\bullet}$, $q : Y^{\bullet} \to X^{\bullet}$ such that $qp = e$ and $pq = 1_{Y^{\bullet}}$. Since every $\mathrm{H}^i(Y^{\bullet})$ is a direct summand of $\mathrm{H}^i(X^{\bullet})$, $Y^{\bullet} \in \mathsf{D}^{\mathrm{b}}_c(\mathsf{Mod}\, A)$. According to Proposition 3.7, we complete the proof. □

## 12. Derived Functors

Throughout this section, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are abelian categories.

### 12.1. **Derived Functors.**

**Definition 12.1.** A triangulated full subcategory $\mathsf{K}^*(\mathcal{A})$ of is called a *quotientizing subcategory* (often called *localizing subcategory*) if the canonical functor

$$\mathsf{K}^*(\mathcal{A})/\mathsf{K}^{*,\phi}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$$

is fully faithful, where $\mathsf{K}^{*,\phi}(\mathcal{A}) = \mathsf{K}^{\phi}(\mathcal{A}) \cap \mathsf{K}^*(\mathcal{A})$. If $\mathsf{K}^*(\mathcal{A})$ is a quotientizing subcategory of $\mathsf{K}(\mathcal{A})$, we denote by $\mathsf{D}^*(\mathcal{A})$ the quotient category $\mathsf{K}^*(\mathcal{A})/\mathsf{K}^{*,\phi}(\mathcal{A})$ and by $Q^*_{\mathcal{A}} : \mathsf{K}^*(\mathcal{A}) \to \mathsf{D}^*(\mathcal{A})$ the canonical quotient functor.

**Definition 12.2** (Right Derived Functor)**.** Let $\mathsf{K}^*(\mathcal{A})$ be a quotientizing subcategory of $\mathsf{K}(\mathcal{A})$ and $F : \mathsf{K}^*(\mathcal{A}) \to \mathsf{K}(\mathcal{B})$ a $\partial$-functor. The *right derived functor* of $F$ is a $\partial$-functor

$$\boldsymbol{R}^* F : \mathsf{D}^*(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$$

together with a functorial morphism of $\partial$-functors

$$\xi \in \partial\,\mathrm{Mor}(Q_{\mathcal{B}}F, \boldsymbol{R}^* F Q^*_{\mathcal{A}})$$

with the following property:
For $G \in \partial(\mathsf{D}^*(\mathcal{A}), \mathsf{D}(\mathcal{B}))$ and $\zeta \in \partial\,\mathrm{Mor}(Q_{\mathcal{B}}F, GQ^*_{\mathcal{A}})$, there exists a unique morphism $\eta \in \partial\,\mathrm{Mor}(\boldsymbol{R}^* F, G)$ such that

$$\zeta = (\eta Q^*_{\mathcal{A}})\xi.$$

In other words, we can simply write the above using functor categories. For triangulated categories $\mathcal{C}, \mathcal{C}'$, the $\partial$-functor category $\partial(\mathcal{C}, \mathcal{C}')$ is the category (?) consisting of $\partial$-functors from $\mathcal{C}$ to $\mathcal{C}'$ as objects and $\partial$-functorial morphisms as morphisms. Then we have

$$\partial\operatorname{Mor}(Q_{\mathcal{B}}F, -Q^*_{\mathcal{A}}) \cong \partial\operatorname{Mor}(\boldsymbol{R}^*F, -)$$

as functors from $\partial(\mathsf{D}^*(\mathcal{A}), \mathsf{D}(\mathcal{B}))$ to $\mathfrak{Set}$ (See Lemma 1.8).

**Definition 12.3.** Let $\mathsf{K}^*(\mathcal{A})$ be a quotientizing subcategory of $\mathsf{K}^*(\mathcal{A})$ and $F : \mathsf{K}^*(\mathcal{A}) \to \mathsf{K}(\mathcal{B})$ a $\partial$-functor. When $F$ has a right derived functor $\boldsymbol{R}^*F : \mathsf{D}^*(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$, we define $\mathrm{R}^i F = \mathrm{H}^i \boldsymbol{R}^*F : \mathsf{D}^*(\mathcal{A}) \to \mathcal{B}$ $(i \in \mathbb{Z})$.

**Proposition 12.4.** *Let $\mathsf{K}^*(\mathcal{A})$ be a quotientizing subcategory of $\mathsf{K}(\mathcal{A})$ and $F : \mathsf{K}^*(\mathcal{A}) \to \mathsf{K}(\mathcal{B})$ a $\partial$-functor. Assume $F$ has a right derived functor $\boldsymbol{R}^*F : \mathsf{D}^*(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$. Then for any exact sequence $O \to X^\cdot \to Y^\cdot \to Z^\cdot \to O$ in $\mathsf{C}(\mathcal{A})$ we have a long exact sequence*

$$\ldots \to \mathrm{R}^i F(X^\cdot) \to \mathrm{R}^i F(Y^\cdot) \to \mathrm{R}^i F(Z^\cdot) \to \mathrm{R}^{i+1} F(X^\cdot) \to \ldots .$$

*Proof.* By Proposition 10.4, it is easy. □

**Theorem 12.5** (Existence Theorem)**.** *Let $\mathsf{K}^*(\mathcal{A})$ be a quotientizing subcategory of $\mathsf{K}(\mathcal{A})$ and $F : \mathsf{K}^*(\mathcal{A}) \to \mathsf{K}(\mathcal{B})$ a $\partial$-functor. Assume there exists a triangulated full subcategory $\mathcal{L}$ of $\mathsf{K}^*(\mathcal{A})$ such that*

(a) *for any $X^\cdot \in \mathsf{K}^*(\mathcal{A})$ there is a quasi-isomorphism $X^\cdot \to I^\cdot$ with $I^\cdot \in \mathcal{L}$,*
(b) *$Q_{\mathcal{B}}F(\mathcal{L}^\phi) = \{O\}$,*

*where $\mathcal{L}^\phi = \mathsf{K}^\phi(\mathcal{A}) \cap \mathcal{L}$. Then there exists the right derived functor $(\boldsymbol{R}^*F, \xi)$ such that $\xi_I : Q_{\mathcal{B}}FI^\cdot \to \boldsymbol{R}^*FI^\cdot$ is a quasi-isomorphism for $I^\cdot \in \mathcal{L}$.*

*Proof.* Let $E : \mathcal{L} \to \mathsf{K}^*(\mathcal{A})$ be the embedding functor, then by the assumption (a) and Proposition 7.16 the canonical functor $\overline{E} : \mathcal{L}/\mathcal{L}^\phi \to \mathsf{D}^*(\mathcal{A})$ is an equivalence. Let $J : \mathsf{D}^*(\mathcal{A}) \to \mathcal{L}/\mathcal{L}^\phi$ be a quasi-inverse of $\overline{E}$. By the assumption (b) and Proposition 8.7 there is a $\partial$-functor $\overline{F} : \mathcal{L}/\mathcal{L}^\phi \to \mathsf{D}(\mathcal{B})$ such that $Q_{\mathcal{B}}FE = \overline{F}Q_{\mathcal{L}}$, where $Q_{\mathcal{L}} : \mathcal{L} \to \mathcal{L}/\mathcal{L}^\phi$ is the canonical quotient. Put $\boldsymbol{R}^*F = \overline{F}J$. Since $Q^*_{\mathcal{A}}E = \overline{E}Q_{\mathcal{L}}$, we have

$$\begin{aligned}\partial\operatorname{Mor}(Q_{\mathcal{B}}FE, GQ^*_{\mathcal{A}}E) &\cong \partial\operatorname{Mor}(\overline{F}Q_{\mathcal{L}}, G\overline{E}Q_{\mathcal{L}}) \\ &\cong \partial\operatorname{Mor}(\overline{F}J\overline{E}, G\overline{E}) \\ &\cong \partial\operatorname{Mor}(\overline{F}J, G).\end{aligned}$$

It remains to show that

$$\partial\operatorname{Mor}(Q_{\mathcal{B}}F, GQ^*_{\mathcal{A}}) \cong \partial\operatorname{Mor}(Q_{\mathcal{B}}FE, GQ^*_{\mathcal{A}}E) \quad (\phi \mapsto \phi E).$$

Let $\phi \in \partial\operatorname{Mor}(Q_{\mathcal{B}}F, GQ^*_{\mathcal{A}})$ with $\phi E = 0$. For any $X^\cdot \in \mathsf{K}^*(\mathcal{A})$ there exists $I^\cdot \in \mathcal{L}$ which has a quasi-isomorphism $s : X^\cdot \to I^\cdot$. Then $\phi_X = (GQ^*_{\mathcal{A}}s)^{-1}\phi_I Q_{\mathcal{B}}Fs = 0$, and hence $\phi = 0$. Given $\psi \in \partial\operatorname{Mor}(Q_{\mathcal{B}}FE, GQ^*_{\mathcal{A}}E)$, for any $X^\cdot \in \mathsf{K}^*(\mathcal{A})$, let $\phi_X = (GQ^*_{\mathcal{A}}s)^{-1}\psi_I Q_{\mathcal{B}}Fs$ for some quasi-isomorphism $s : X^\cdot \to I^\cdot$, with $I^\cdot \in \mathcal{L}$. For another quasi-isomorphism $s' : X^\cdot \to I'^\cdot$, by the assumption (a), we have a commutative diagram

$$\begin{array}{ccc} X^\cdot & \xrightarrow{\;s\;} & I^\cdot \\ {\scriptstyle s'}\big\downarrow & & \big\downarrow{\scriptstyle t'} \\ I'^\cdot & \xrightarrow{\;t\;} & I''^\cdot \end{array}$$

where all morphisms are quasi-isomorphisms and $I''^{\cdot} \in \mathcal{L}$. Then we have

$$\begin{aligned}(GQ^*_{\mathcal{A}}s)^{-1}\psi_I Q_{\mathcal{B}}Fs &= (GQ^*_{\mathcal{A}}t's)^{-1}\psi_{I''}Q_{\mathcal{B}}Ft's \\ &= (GQ^*_{\mathcal{A}}ts')^{-1}\psi_{I''}Q_{\mathcal{B}}Fts' \\ &= (GQ^*_{\mathcal{A}}s')^{-1}\psi_{I'}Q_{\mathcal{B}}Fs'.\end{aligned}$$

It is not hard to see that $\phi \in \partial\,\mathrm{Mor}(Q_{\mathcal{B}}F, GQ^*_{\mathcal{A}})$. The last assertion is easy to check. □

**Corollary 12.6.** *Assume that there exists an additive subcategory $\mathcal{I}$ of $\mathcal{A}$ such that*

(a) *every $X \in \mathcal{A}$ has a monomorphism to an object in $\mathcal{I}$.*
(b) *for an exact sequence $O \to X \to Y \to Z \to O$ in $\mathcal{A}$ with $X \in \mathcal{I}$, $Y \in \mathcal{I}$ if and only if $Z \in \mathcal{I}$,*
(c) *for an exact sequence $O \to X \to Y \to Z \to O$ in $\mathcal{A}$ with $X, Y, Z \in \mathcal{I}$, then $O \to FX \to FY \to FZ \to O$ is exact in $\mathcal{B}$.*

*For any additive functor $F : \mathcal{A} \to \mathcal{B}$, $\boldsymbol{R}^+F : \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ exists.*

*Proof.* Let $\mathcal{L} = \mathsf{K}^+(\mathcal{I})$, then it is easy to see that $\mathcal{L}$ satisfies the conditions of Theorem 12.5. □

**Proposition 12.7.** *Let $\mathsf{K}^{**}(\mathcal{A}) \subset \mathsf{K}^*(\mathcal{A})$ be quotientizing subcategories of $\mathsf{K}(\mathcal{A})$ and $F : \mathsf{K}^*(\mathcal{A}) \to \mathsf{K}(\mathcal{B})$ a $\partial$-functor. Assume $\mathsf{K}^*(\mathcal{A})$ has a triangulated full subcategory $\mathcal{L}$ such that*

(a) *for any $X^{\cdot} \in \mathsf{K}^*(\mathcal{A})$, there exists a quasi-isomorphism $X^{\cdot} \to I^{\cdot}$ with $I^{\cdot} \in \mathcal{L}$,*
(b) *for any $X^{\cdot} \in \mathsf{K}^{**}(\mathcal{A})$, there exists a quasi-isomorphism $X^{\cdot} \to I^{\cdot}$ with $I^{\cdot} \in \mathcal{L} \cap \mathsf{K}^{**}(\mathcal{A})$, and*
(c) *$Q_{\mathcal{B}}F(\mathcal{L}^{\phi}) = \{O\}$,*

*where $\mathcal{L}^{\phi} = \mathsf{K}^{\phi}(\mathcal{A}) \cap \mathcal{L}$. Then both $F$ and $F|_{\mathsf{K}^{**}(\mathcal{A})}$ have the right derived functors $(\boldsymbol{R}^*F, \xi)$ and $(\boldsymbol{R}^{**}(F|_{\mathsf{K}^{**}(\mathcal{A})}), \zeta)$, respectively, and the canonical $\partial$-functorial morphism*

$$\phi : \boldsymbol{R}^{**}(F|_{\mathsf{K}^{**}(\mathcal{A})}) \to \boldsymbol{R}^*F|_{\mathsf{K}^{**}(\mathcal{A})}$$

*is an isomorphism.*

*Proof.* By Theorem 12.5 both $F$ and $F|_{\mathsf{K}^{**}(\mathcal{A})}$ have the right derived functors $(\boldsymbol{R}^*F, \xi)$ and $(\boldsymbol{R}^{**}(F|_{\mathsf{K}^{**}(\mathcal{A})}), \zeta)$, respectively, and we have a unique $\partial$-functorial morphism

$$\phi : \boldsymbol{R}^{**}(F|_{\mathsf{K}^{**}(\mathcal{A})}) \to \boldsymbol{R}^*F|_{\mathsf{K}^{**}(\mathcal{A})}$$

such that $\xi|_{\mathsf{K}^{**}(\mathcal{A})} = (\phi Q^{**}_{\mathcal{A}})\zeta$. For any $I^{\cdot} \in \mathcal{L} \cap \mathsf{K}^{**}(\mathcal{A})$, by Theorem 12.5 both $\xi_I$ and $\zeta_I$ are isomorphisms, so that $\phi_{QI}$ is an isomorphism. Thus, since by assumption (b), the canonical functor $Q : \mathcal{L} \cap \mathsf{K}^{**}(\mathcal{A}) \to \mathsf{D}^{**}(\mathcal{A})$ is dense, $\phi$ is an isomorphism. □

**Proposition 12.8.** *Let $\mathsf{K}^*(\mathcal{A})$ be a quotientizing subcategory of $\mathsf{K}(\mathcal{A})$ and $F : \mathsf{K}^*(\mathcal{A}) \to \mathsf{K}(\mathcal{B})$ a $\partial$-functor. Let $\mathsf{K}^{\dagger}(\mathcal{B})$ be a quotientizing subcategory of $\mathsf{K}(\mathcal{B})$ and $G : \mathsf{K}^{\dagger}(\mathcal{B}) \to \mathsf{K}(\mathcal{C})$ a $\partial$-functor. Assume*

(a) *$\mathsf{K}^*(\mathcal{A})$ has a triangulated full subcategory $\mathcal{L}$ for which the assumptions 1, 2 of Theorem 12.5 are satisfied,*
(b) *$\mathsf{K}^{\dagger}(\mathcal{B})$ has a triangulated full subcategory $\mathcal{M}$ for which the assumptions 1, 2 of Theorem 12.5 are satisfied, and*
(c) *$F(\mathsf{K}^*(\mathcal{A})) \subset \mathsf{K}^{\dagger}(\mathcal{B})$ and $F(\mathcal{L}) \subset \mathcal{M}$.*

*Then $F, G$ and $GF$ have the right derived functors $(\boldsymbol{R}^*F, \xi)$, $(\boldsymbol{R}^\dagger G, \zeta)$ and $(\boldsymbol{R}^*(GF), \eta)$ with $\boldsymbol{R}^*F(\mathsf{D}^*(\mathcal{A})) \subset \mathsf{C}^\dagger(\mathcal{B})$, and the canonical homomorphism*

$$\phi : \boldsymbol{R}^*(GF) \to \boldsymbol{R}^\dagger G \circ \boldsymbol{R}^*F$$

*is an isomorphism.*

*Proof.* By Theorem 12.5 $F$ and $G$ have the right derived functors $(\boldsymbol{R}^*F, \xi)$ and $(\boldsymbol{R}^\dagger G, \zeta)$, respectively. Let $X^\cdot \in \mathcal{L}$ be acyclic. Then, since $Q(F(X^\cdot)) = 0$, $F(X^\cdot)$ is acyclic and $Q(G(F(X^\cdot))) = 0$. Thus, again by Theorem 12.5 $GF$ has a right derived functor $(\boldsymbol{R}^*(GF), \eta)$. Also, for any $X^\cdot \in \mathsf{D}^*(\mathcal{A})$, since we have a quasi-isomorphism $X^\cdot \to I^\cdot$ with $I^\cdot \in \mathcal{L}$, $\boldsymbol{R}^*F(X^\cdot) \cong \boldsymbol{R}^*F(Q(I^\cdot)) \cong Q(F(I^\cdot)) \in \mathsf{D}^\dagger(\mathcal{B})$. Thus by Theorem 12.5 we have a unique homomorphism of $\partial$-functors

$$\phi : \boldsymbol{R}^*(GF) \to \boldsymbol{R}^\dagger G \circ \boldsymbol{R}^*F$$

such that $(\boldsymbol{R}^\dagger G\xi)(\zeta F) = (\phi Q)\eta$. Let $I^\cdot \in \mathcal{L}$. Then $\xi_I, \zeta_{FI}$ and $\eta_I$ are isomorphisms, so that $\phi_{QI}$ is an isomorphism. Thus $\phi$ is an isomorphism, because $Q : \mathcal{L} \to \mathsf{D}^*(\mathcal{A})$ is dense □

12.2. **Way-out Functors.**

**Definition 12.9** (Way-out Functor)**.** Let $\mathcal{A}, \mathcal{B}$ be abelian categories, $\mathcal{A}'$ a thick abelian full subcategory of $\mathcal{A}$. Let $\mathsf{K}^*(\mathcal{A})$ be a quotientizing subcategory of $\mathsf{K}(\mathcal{A})$. A $\partial$-functor $F : \mathsf{D}^*_{\mathcal{A}'}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ is called *way-out right* (resp., *way-out left*) provided that for any $n_1 \in \mathbb{Z}$ there exists $n_2 \in \mathbb{Z}$ such that if $X^\cdot \in \mathsf{D}^*(\mathcal{A})$ is a complex with $\mathrm{H}^i(X^\cdot) = O$ for all $i < n_2$ (resp., $i > n_2$), then $\mathrm{H}^i(FX^\cdot) = O$ for all $i < n_1$ (resp., $i > n_1$).

Moreover, $F$ is called *way-out in both directions* if $F$ is way-out right and way-out left.

**Lemma 12.10.** *For a complex $X^\cdot \in \mathsf{C}(\mathcal{A})$, we have triangles in $\mathsf{D}(\mathcal{A})$*

1.
$$\tau_{\geq n-1}X^\cdot \to \tau_{\geq n}X^\cdot \to X^n[-n] \to \tau_{\geq n-1}X^\cdot[1].$$

2.
$$\mathrm{H}^n(X^\cdot)[-n] \to \sigma_{>n-1}X^\cdot \to \sigma_{>n}X^\cdot \to \mathrm{H}^n(X^\cdot)[1-n].$$

*Proof.* By 10.6, we have an exact sequence

$$O \to Y^\cdot \to \sigma_{>n-1}X^\cdot \to \sigma_{>n}X^\cdot \to O,$$

where $Y^\cdot : Y^{n-1} \to Y^n = \operatorname{Im} d_X^{n-1} \to \operatorname{Ker} d_X^n$. Then it is easy to see $Y^\cdot \cong \mathrm{H}^n(X^\cdot)[-n]$. □

**Proposition 12.11.** *Let $\mathcal{A}, \mathcal{B}$ be abelian categories, $\mathcal{A}'$ a thick abelian full subcategory of $\mathcal{A}$. Let $F^*, G^* : \mathsf{D}^*_{\mathcal{A}'}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ be $\partial$-functors, and $\eta^* \in \partial\operatorname{Mor}(F^*, G^*)$, where $* =$ nothing, $+, -, b$. Then the following hold.*

1. *If $\eta^{\mathrm{b}}(X)$ are isomorphisms for all $X \in \mathcal{A}'$, then $\eta^{\mathrm{b}}$ is an isomorphism.*
2. *Assume that $F$ and $G$ are way-out right. If $\eta^+(X)$ are isomorphisms for all $X \in \mathcal{A}'$, then $\eta^+$ is an isomorphism.*
3. *Assume that $F$ and $G$ are way-out in both directions. If $\eta(X)$ are isomorphisms for all $X \in \mathcal{A}'$, then $\eta$ is an isomorphism.*

4. *Let $\mathcal{I}$ be a collection of objects of $\mathcal{A}'$ such that every $X \in \mathcal{A}'$ admits a monomorphism to an object of $\mathcal{I}$. Assume that $F$ and $G$ are way-out right. If $\eta^*(I)$ are isomorphisms for all $I \in \mathcal{I}$, then $\eta^*(X)$ are isomorphisms for all $X \in \mathcal{A}'$.*

*Proof.* 1. Let $X^\cdot \in \mathsf{D}^{\mathrm{b}}(\mathcal{A})$. For $n \gg 0$, $\sigma_{>n}X^\cdot = O$, and then $\eta(\sigma_{>n}X^\cdot)$ is an isomorphism. By Lemma 12.10, $\eta(\sigma_{>n-1}X^\cdot)$ is an isomorphism. Since $X^\cdot \in \mathsf{D}^{\mathrm{b}}(\mathcal{A})$, we get the statement.

2. Let $X^\cdot \in \mathsf{D}^{+}(\mathcal{A})$. For any $n \in \mathbb{Z}$, we show that $\mathrm{H}^n(\eta(X^\cdot))$ is an isomorphism. Put $n_1 = n+1$. Then there exists $n_2 \in \mathbb{Z}$ such that if $Y^\cdot \in \mathsf{D}^*(\mathcal{A})$ is a complex with $\mathrm{H}^i(Y^\cdot) = O$ for all $i < n_2$, then $\mathrm{H}^i(FY^\cdot) = O$ and $\mathrm{H}^i(GY^\cdot) = O$ for all $i < n_1$. Since $\mathrm{H}^i(\sigma_{>n_2}X^\cdot) = O$ for $i < n_1$, we have

$$\begin{aligned}\mathrm{H}^n(F\sigma_{>n_2}X^\cdot) &= \mathrm{H}^{n-1}(F\sigma_{>n_2}X^\cdot) = O,\\ \mathrm{H}^n(G\sigma_{>n_2}X^\cdot) &= \mathrm{H}^{n-1}(G\sigma_{>n_2}X^\cdot) = O.\end{aligned}$$

Considering a triangle $\sigma_{\le n_2}X^\cdot \to X^\cdot \to \sigma_{>n_2}X^\cdot \to \sigma_{\le n_2}X^\cdot[1]$, we have a commutative diagram

$$\begin{array}{ccc}\mathrm{H}^n(F\sigma_{\le n_2}X^\cdot) & \xrightarrow{\sim} & \mathrm{H}^n(FX^\cdot)\\ \wr\downarrow & & \downarrow\\ \mathrm{H}^n(G\sigma_{\le n_2}X^\cdot) & \xrightarrow{\sim} & \mathrm{H}^n(GX^\cdot).\end{array}$$

where all horizontal arrows are isomorphisms. By 1, the left vertical arrow are an isomorphism, and hence $\eta(\mathrm{H}^n(X^\cdot))$ is an isomorphism.

3. As in 2, for any $X^\cdot \in \mathsf{D}(\mathcal{A})$, $\eta(\sigma_{>0}X^\cdot)$ is an isomorphism. Considering $\sigma_{\le 0}X^\cdot \to X^\cdot \to \sigma_{>0}X^\cdot \to \sigma_{\le 0}X^\cdot[1]$, $\eta(X^\cdot)$ is an isomorphism.

4. For $X \in \mathcal{A}'$, by the dual of Lemma 10.10, there is a resolution $I^\cdot$ with each $I^i \in \mathcal{I}$. By replacing $\sigma$ by $\tau$ in 1 and 2, we have the statement. □

**Proposition 12.12.** *Let $\mathcal{A}, \mathcal{B}$ be abelian categories, $\mathcal{A}', \mathcal{B}'$ thick abelian full subcategories of $\mathcal{A}, \mathcal{B}$, respectively. Let $F^* : \mathsf{D}^*_{\mathcal{A}'}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ be a $\partial$-functor, where $* =$ nothing, $+, -, b$. Then the following hold.*

1. *If $F^{\mathrm{b}}(X) \in \mathsf{D}_{\mathcal{B}'}(\mathcal{B})$ for all $X \in \mathcal{A}'$, then $F^{\mathrm{b}}(X) \in \mathsf{D}_{\mathcal{B}'}(\mathcal{B})$ for all $X^\cdot \in \mathsf{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$.*
2. *Assume that $F$ and $G$ are way-out right. If $F^{\mathrm{b}}(X) \in \mathsf{D}_{\mathcal{B}'}(\mathcal{B})$ for all $X \in \mathcal{A}'$, then $F^{\mathrm{b}}(X) \in \mathsf{D}_{\mathcal{B}'}(\mathcal{B})$ for all $X^\cdot \in \mathsf{D}^{+}_{\mathcal{A}'}(\mathcal{A})$.*
3. *Assume that $F$ and $G$ are way-out in both directions. If $F^{\mathrm{b}}(X) \in \mathsf{D}_{\mathcal{B}'}(\mathcal{B})$ for all $X \in \mathcal{A}'$, then $F^{\mathrm{b}}(X) \in \mathsf{D}_{\mathcal{B}'}(\mathcal{B})$ for all $X^\cdot \in \mathsf{D}_{\mathcal{A}'}(\mathcal{A})$.*
4. *Let $\mathcal{I}$ be a collection of objects of $\mathcal{A}'$ such that every $X \in \mathcal{A}'$ admits a monomorphism to an object of $\mathcal{I}$. Assume that $F$ and $G$ are way-out right. If $F(I) \in \mathsf{D}_{\mathcal{B}'}(\mathcal{B})$ for all $I \in \mathcal{I}$, then $F(X) \in \mathsf{D}_{\mathcal{B}'}(\mathcal{B})$ for all $X \in \mathcal{A}'$.*

*Proof.* The same as the proof of Proposition 12.12 (left to the reader). □

## 13. Double Complexes

Throughout this section, $\mathcal{A}$ is an abelian category.

**Definition 13.1** (Double Complex). A *double complex* $C^{\cdot\cdot}$ is a bigraded object $(C^{p,q})_{p,q\in\mathbb{Z}}$ of $\mathcal{A}$ together with $d_{\mathrm{I}}^{p,q} : C^{p,q} \to C^{p+1,q}$ and $d_{\mathrm{II}}^{p,q} : C^{p,q} \to C^{p,q+1}$ such that

$$C^{\cdot q} = (C^{p,q}, d_{\mathrm{I}}^{p,q} : C^{p,q} \to C^{p+1,q})$$
$$C^{p\cdot} = (C^{p,q}, d_{\mathrm{II}}^{p,q} : C^{p,q} \to C^{p,q+1})$$

are complexes satisfying $d_{\mathrm{I}}^{p,q+1} d_{\mathrm{II}}^{p,q} + d_{\mathrm{II}}^{p+1,q} d_{\mathrm{I}}^{p,q} = 0$.

A *morphism $f$ of double complexes* $X^{\cdot\cdot}$ to $Y^{\cdot\cdot}$ is a collection of morphisms $f^{p,q} : X^{p,q} \to Y^{p,q}$ such that $f^{\cdot q} : X^{\cdot q} \to Y^{\cdot q}$ and $f^{p\cdot} : X^{p\cdot} \to Y^{p\cdot}$ are morphisms of complexes for all $p, q \in \mathbb{Z}$.

We denote by $\mathsf{C}^2(\mathcal{A})$ the categories of double complexes of $\mathcal{A}$. Auto-equivalences $T_{\mathrm{I}}, T_{\mathrm{II}} : \mathsf{C}^2(\mathcal{A}) \to \mathsf{C}^2(\mathcal{A})$ are called the translations if $(T_{\mathrm{I}} X^{\cdot\cdot})^{p,q} = X^{p+1,q}$ and $(T_{\mathrm{I}} d_{\# X})^{p,q}$
$= -d_{\# X}^{p+1,q}$ and $(T_{\mathrm{II}} X^{\cdot\cdot})^{p,q} = X^{p,q+1}$ and $(T_{\mathrm{II}} d_{\# X})^{p,q} = -d_{\# X}^{p,q+1}$ for any complex $X^{\cdot\cdot} = (X^{p,q}, d_{\mathrm{I}\,X}^{p,q}, d_{\mathrm{II}\,X}^{p,q})$, where $\# = I, II$.

Moreover, an *$r$-tuple complex* $C^{\cdot^r}$ is an $r$-tuple graded object $(C^{\boldsymbol{p}})_{\boldsymbol{p} \in \mathbb{Z}^r}$ of $\mathcal{A}$ together with $d_i^{\boldsymbol{p}} : C^{\boldsymbol{p}} \to C^{\boldsymbol{p}+\boldsymbol{e}_i}$ $(1 \le i \le r)$ such that

$$d_i^2 = 0 \ (1 \le i \le r),$$
$$d_i d_j + d_j d_i = 0 \text{ for all } i, j,$$

where $\boldsymbol{e}_i = (0, \dots, 0, \underset{i\text{th component}}{1}, 0, \dots, 0)$.

**Proposition 13.2.** *Let $\mathsf{C}_{\mathrm{I}}(\mathcal{A})$ (resp., $\mathsf{C}_{\mathrm{II}}(\mathcal{A})$) be the full subcategory of $\mathsf{C}^2(\mathcal{A})$ consisting of complexes $X^{\cdot\cdot}$ such that $X^{p,q} = O$ for all $q \neq 0$ (resp., $p \neq 0$). Then we have $\mathsf{C}(\mathcal{A}) \cong \mathsf{C}_{\mathrm{I}}(\mathcal{A}) \cong \mathsf{C}_{\mathrm{II}}(\mathcal{A})$ and $\mathsf{C}^2(\mathcal{A}) \cong \mathsf{C}(\mathsf{C}_{\mathrm{I}}(\mathcal{A})) \cong \mathsf{C}(\mathsf{C}_{\mathrm{II}}(\mathcal{A}))$. In particular, $\mathsf{C}^2(\mathcal{A})$ is an abelian category.*

*Proof.* We define a functor $F_{\mathrm{I}} : \mathsf{C}^2(\mathcal{A}) \to \mathsf{C}(\mathsf{C}_{\mathrm{I}}(\mathcal{A}))$ as follows. For any double complex $X^{\cdot\cdot} = (X^{p,q}, d_{\mathrm{I}\,X}^{p,q}, d_{\mathrm{II}\,X}^{p,q})$, $F_{\mathrm{I}}(X^{p,q}, d_{\mathrm{I}\,X}^{p,q}, d_{\mathrm{II}\,X}^{p,q}) = (X^{\cdot,q}, d_{\mathrm{II}\,X}^{\cdot,q})$ where $X^{\cdot,q} = (X^{p,q}, (-1)^q d_{\mathrm{I}\,X}^{p,q})$. For a morphism $f : X^{\cdot\cdot} \to Y^{\cdot\cdot}$, $F_{\mathrm{I}}(f)^{p,q} = f^{p,q}$. Then it is easy to see that $F_{\mathrm{I}}$ is an equivalence. $\square$

By the above, we can deal with $\mathsf{C}^2(\mathcal{A})$ as $\mathsf{C}(\mathsf{C}_{\mathrm{I}}(\mathcal{A}))$ or $\mathsf{C}(\mathsf{C}_{\mathrm{II}}(\mathcal{A}))$.

**Definition 13.3** (Truncations)**.** For a double complex $X^{\cdot\cdot} = (X^{p,q}, d_{\mathrm{I}}^{p,q}, d_{\mathrm{II}}^{p,q})$, we define the following truncations:

$$(\sigma_{>n}^{\mathrm{I}} X^{\cdot\cdot})^{p,q} = \begin{cases} O \text{ if } p < n \\ \operatorname{Im} d_{\mathrm{I}}^{p,q} \text{ if } p = n \\ X^{p,q} \text{ if } p > n \end{cases} \quad (\sigma_{\le n}^{\mathrm{I}} X^{\cdot\cdot})^{p,q} = \begin{cases} X^{p,q} \text{ if } p < n \\ \operatorname{Ker} d_{\mathrm{I}}^{p,q} \text{ if } p = n \\ O \text{ if } p > n \end{cases}$$

$$(\sigma_{>n}^{\mathrm{II}} X^{\cdot\cdot})^{p,q} = \begin{cases} O \text{ if } q < n \\ \operatorname{Im} d_{\mathrm{II}}^{p,q} \text{ if } q = n \\ X^{p,q} \text{ if } q > n \end{cases} \quad (\sigma_{\le n}^{\mathrm{II}} X^{\cdot\cdot})^{p,q} = \begin{cases} X^{p,q} \text{ if } q < n \\ \operatorname{Ker} d_{\mathrm{II}}^{p,q} \text{ if } q = n \\ O \text{ if } q > n \end{cases}$$

$$(\tau_{\le n}^{\mathrm{I}} X^{\cdot\cdot})^{p,q} = \begin{cases} X^{p,q} \text{ if } p \le n \\ O \text{ if } p > n \end{cases} \quad (\tau_{\ge n}^{\mathrm{I}} X^{\cdot\cdot})^{p,q} = \begin{cases} O \text{ if } p < n \\ X^{p,q} \text{ if } p \ge n \end{cases}$$

$$(\tau_{\le n}^{\mathrm{II}} X^{\cdot\cdot})^{p,q} = \begin{cases} X^{p,q} \text{ if } q \le n \\ O \text{ if } q > n \end{cases} \quad (\tau_{\ge n}^{\mathrm{II}} X^{\cdot\cdot})^{p,q} = \begin{cases} O \text{ if } q < n \\ X^{p,q} \text{ if } q \ge n \end{cases}$$

**Lemma 13.4.** *For a double complex $X^{\cdot\cdot} = (X^{p,q}, d_{\mathrm{I}}, d_{\mathrm{II}})$, the following hold.*

1. *We have exact sequences in* $\mathsf{C}^2(\mathcal{A})$
$$O \to \sigma^{\#}_{\le n} X^{\cdot\cdot} \to X^{\cdot\cdot} \to \sigma^{\#}_{>n} X^{\cdot\cdot} \to O$$
*for all* $n \in \mathbb{Z}$ *with* $\# = I, II$.
2. *We have exact sequences in* $\mathsf{C}^2(\mathcal{A})$
$$O \to \tau^{\#}_{\ge n} X^{\cdot\cdot} \to X^{\cdot\cdot} \to \tau^{\#}_{\le n-1} X^{\cdot\cdot} \to O$$
*for all* $n \in \mathbb{Z}$ *with* $\# = I, II$.

**Definition 13.5** (Total Complexes)**.** For a double complex $X^{\cdot\cdot} = (X^{p,q}, d_{\mathrm{I}}^{p,q}, d_{\mathrm{II}}^{p,q})$, we define the *total complexes*
$$\operatorname{Tot} C^{\cdot\cdot} = (X^n, d^n), \text{where } X^n = \coprod\nolimits_{p+q=n} C^{p,q}, d^n = \coprod\nolimits_{p+q=n} d_{\mathrm{I}}^{p,q} + d_{\mathrm{II}}^{p,q}$$
$$\overset{\wedge}{\operatorname{Tot}} C^{\cdot\cdot} = (Y^n, d^n), \text{where } Y^n = \prod\nolimits_{p+q=n} C^{p,q}, d^n = \prod\nolimits_{p+q=n} d_{\mathrm{I}}^{p,q} + d_{\mathrm{II}}^{p,q}.$$

Moreover, for an $r$-tuple complex $X^{\cdot^r} = (X^{\boldsymbol{p}}, d_i^{\boldsymbol{p}})$ $(1 \le i \le r)$, we define the *total complexes*
$$\operatorname{Tot} C^{\cdot^r} = (X^n, d^n), \text{where } X^n = \coprod\nolimits_{|\boldsymbol{p}|=n} C^{\boldsymbol{p}}, d^n = \coprod\nolimits_{|\boldsymbol{p}|=n} \sum_{i=1}^{r} d_i^{\boldsymbol{p}}$$
$$\overset{\wedge}{\operatorname{Tot}} C^{\cdot^r} = (Y^n, d^n), \text{where } Y^n = \prod\nolimits_{|\boldsymbol{p}|=n} C^{\boldsymbol{p}}, d^n = \prod\nolimits_{|\boldsymbol{p}|=n} \sum_{i=1}^{r} d_i^{\boldsymbol{p}},$$
where $|\boldsymbol{p}| = |(p_1, \dots, p_r)| = p_1 + \ldots + p_r$.

**Lemma 13.6.** *The following hold.*
1. *If* $\mathcal{A}$ *satisfies the condition Ab4, then the functor* $\operatorname{Tot} : \mathsf{C}^2(\mathcal{A}) \to \mathsf{C}(\mathcal{A})$ *preserves translations, exact sequences and coproducts.*
2. *If* $\mathcal{A}$ *satisfies the condition Ab4*, then the functor* $\overset{\wedge}{\operatorname{Tot}} : \mathsf{C}^2(\mathcal{A}) \to \mathsf{C}(\mathcal{A})$ *preserves translations, exact sequences and products.*

**Lemma 13.7.** *For a double complex* $X^{\cdot\cdot} = (X^{p,q}, d_{\mathrm{I}}, d_{\mathrm{II}})$*, the following hold.*
1. *If* $X^{p,q} = O$ *for* $q < m, n < q$ *for* $m \le n$ *and* $X^{\cdot q}$ *are acyclic complexes in* $\mathsf{C}(\mathcal{A})$ *for all* $q$*, then* $\operatorname{Tot} X^{\cdot\cdot}$ *is acyclic in* $\mathsf{C}(\mathcal{A})$ *.*
2. *If* $X^{p,q} = O$ *for* $q < m, n < q$ *for* $m \le n$ *and* $X^{p\cdot}$ *are acyclic complexes in* $\mathsf{C}(\mathcal{A})$ *for all* $p$*, then* $\operatorname{Tot} X^{\cdot\cdot}$ *is acyclic in* $\mathsf{C}(\mathcal{A})$ *.*
3. *Assume* $\mathcal{A}$ *satisfies the condition Ab4. If* $X^{p,q} = O$ *for* $q > n$ *and* $X^{\cdot q}$ *are acyclic complexes in* $\mathsf{C}(\mathcal{A})$ *for all* $q$*, then* $\operatorname{Tot} X^{\cdot\cdot}$ *is acyclic in* $\mathsf{C}(\mathcal{A})$ *.*
4. *Assume* $\mathcal{A}$ *satisfies the condition Ab4*. If* $X^{p,q} = O$ *for* $q < n$ *and* $X^{\cdot q}$ *are acyclic complexes in* $\mathsf{C}(\mathcal{A})$ *for all* $q$*, then* $\overset{\wedge}{\operatorname{Tot}} X^{\cdot\cdot}$ *is acyclic in* $\mathsf{C}(\mathcal{A})$ *.*
5. *Assume* $\mathcal{A}$ *satisfies the condition Ab5. If* $X^{p,q} = O$ *for* $q < n$ *and* $X^{p\cdot}$ *are acyclic complexes in* $\mathsf{C}(\mathcal{A})$ *for all* $p$*, then* $\operatorname{Tot} X^{\cdot\cdot}$ *is acyclic in* $\mathsf{C}(\mathcal{A})$ *.*
6. *Assume* $\mathcal{A} = \mathsf{Mod}\, A$ *for a ring* $A$*. If* $X^{p,q} = O$ *for* $q > n$ *and* $X^{p\cdot}$ *are acyclic complexes in* $\mathsf{C}(\mathcal{A})$ *for all* $p$*, then* $\overset{\wedge}{\operatorname{Tot}} X^{\cdot\cdot}$ *is acyclic in* $\mathsf{C}(\mathcal{A})$ *.*

*Proof.* 1. Let $n_X = n - m$. By Lemma 13.4, we have an exact sequence in $\mathsf{C}^2(\mathcal{A})$
$$O \to \tau^{\mathrm{II}}_{\ge n-1} X^{\cdot\cdot} \to X^{\cdot\cdot} \to \tau^{\mathrm{II}}_{\le n} X^{\cdot\cdot} \to O.$$
Then by Lemma 13.6, we have the exact sequence in $\mathsf{C}(\mathcal{A})$
$$O \to \operatorname{Tot} \tau^{\mathrm{II}}_{\ge n-1} X^{\cdot\cdot} \to \operatorname{Tot} X^{\cdot\cdot} \to X^{\cdot n}[-n] \to O.$$

By the assumption of induction on $n_X$, $\operatorname{Tot}\tau^{\mathrm{II}}_{\geq n-1}X^{\cdot\cdot}$ is acyclic. Then $\operatorname{Tot}X^{\cdot\cdot}$ is acyclic because $X^{\cdot n}[-n]$ is acyclic.

2. Let $n_X = n-m$. By Lemma 13.4, we have an exact sequence in $\mathsf{C}^2(\mathcal{A})$

$$O \to \sigma^{\mathrm{II}}_{\leq n-1}X^{\cdot\cdot} \to X^{\cdot\cdot} \to \sigma^{\mathrm{II}}_{>n-1}X^{\cdot\cdot} \to O.$$

Then by Lemma 13.6, we have the exact sequence in $\mathsf{C}^2(\mathcal{A})$

$$O \to \operatorname{Tot}\sigma^{\mathrm{II}}_{\leq n-1}X^{\cdot\cdot} \to \operatorname{Tot}X^{\cdot\cdot} \to \operatorname{Tot}\sigma^{\mathrm{II}}_{>n-1}X^{\cdot n} \to O.$$

By the assumption of induction on $n_X$, $\operatorname{Tot}\sigma^{\mathrm{II}}_{\leq n-1}X^{\cdot\cdot}$ is acyclic. It is easy to see that $\operatorname{Tot}\sigma^{\mathrm{II}}_{>n-1}X^{\cdot n} \cong \mathrm{M}^{\cdot}(1_{X^{\cdot n}})[-n]$ is acyclic. Then $\operatorname{Tot}X^{\cdot\cdot}$ is acyclic.

3. By Lemma 13.4, we have the canonical morphisms in $\mathsf{C}^2(\mathcal{A})$

$$\tau^{\mathrm{II}}_{\geq -r}X^{\cdot\cdot} \xrightarrow{f_r} \tau^{\mathrm{II}}_{\geq -(r+1)}X^{\cdot\cdot},$$

which are term-split monomorphisms for all $p,q$. Since $\operatorname{Tot}f_r : \operatorname{Tot}\tau^{\mathrm{II}}_{\geq -r}X^{\cdot\cdot} \to \operatorname{Tot}\tau^{\mathrm{II}}_{\geq -(r+1)}X^{\cdot\cdot}$ is term-split monomorphisms in $\mathcal{A}$, by Proposition 11.7, we have

$$\underrightarrow{\mathrm{hlim}}\ \operatorname{Tot}\tau^{\mathrm{II}}_{\geq -r}X^{\cdot\cdot} \cong \varinjlim \operatorname{Tot}\tau^{\mathrm{II}}_{\geq -r}X^{\cdot\cdot} \cong \operatorname{Tot}X^{\cdot\cdot}.$$

By 1, $\operatorname{Tot}\tau^{\mathrm{II}}_{\geq -r}X^{\cdot\cdot}$ is acyclic for all $r$. Then $\underrightarrow{\mathrm{hlim}}\ \operatorname{Tot}\tau^{\mathrm{II}}_{\geq -r}X^{\cdot\cdot}$ is acyclic, and hence so is $\operatorname{Tot}X^{\cdot\cdot}$.

4. Dual of 3.

5. By Lemma 13.4, we have the canonical morphisms in $\mathsf{C}^2(\mathcal{A})$

$$\sigma^{\mathrm{II}}_{\leq r}X^{\cdot\cdot} \xrightarrow{g_r} \sigma^{\mathrm{II}}_{\leq r+1}X^{\cdot\cdot}.$$

By Exercise 11.5, we have an exact sequence in $\mathsf{C}(\mathcal{A})$

$$O \to \coprod_r \operatorname{Tot}\sigma^{\mathrm{II}}_{\leq r}X^{\cdot\cdot} \to \coprod_r \operatorname{Tot}\sigma^{\mathrm{II}}_{\leq r}X^{\cdot\cdot} \to \varinjlim \operatorname{Tot}\sigma^{\mathrm{II}}_{\leq r}X^{\cdot\cdot} \to O.$$

Then we have

$$\underrightarrow{\mathrm{hlim}}\operatorname{Tot}\sigma^{\mathrm{II}}_{\leq r}X^{\cdot\cdot} \cong \varinjlim \operatorname{Tot}\sigma^{\mathrm{II}}_{\leq r}X^{\cdot\cdot} \cong \operatorname{Tot}X^{\cdot\cdot}.$$

By 2, $\operatorname{Tot}\sigma^{\mathrm{II}}_{\leq r}X^{\cdot\cdot}$ is acyclic for all $r$. Then $\underrightarrow{\mathrm{hlim}}\ \operatorname{Tot}\sigma^{\mathrm{II}}_{\leq r}X^{\cdot\cdot}$ is acyclic, and hence so is $\operatorname{Tot}X^{\cdot\cdot}$.

6. Dual of 5. □

**Definition 13.8.** We define an embedding functor $em^{\mathrm{I}} : \mathsf{C}(\mathcal{A}) \to \mathsf{C}^2(\mathcal{A})$ as follows. For a complex $X^{\cdot} \in \mathsf{C}(\mathcal{A})$,

$$em^{\mathrm{I}}(X^{\cdot})^{p,q} = \begin{cases} X^p \text{ if } q = 0 \\ O \text{ othewise.} \end{cases}$$

**Definition 13.9** (Proper Exact)**.** An exact sequence $O \to X^{\cdot} \xrightarrow{f} Y^{\cdot} \xrightarrow{g} Z^{\cdot} \to O$ in $\mathsf{C}(\mathcal{A})$ is called *proper exact* if the induced sequence $O \to \mathrm{Z}^{\cdot}(X^{\cdot}) \to \mathrm{Z}^{\cdot}(Y^{\cdot}) \to \mathrm{Z}^{\cdot}(Z^{\cdot}) \to O$ is also exact. In this case, $f$ (resp., $g$) is called a *proper monomorphism* (resp., a *proper epimorphism*).

A complex $X^{\cdot} \in \mathsf{C}(\mathcal{A})$ is called a *proper projective complex* (resp., a *proper injective complex*) if

$$X^{\cdot} \cong \bigoplus_{n\in\mathbb{Z}} P^n[-n] \oplus \bigoplus_{n\in\mathbb{Z}} \mathrm{M}^{\cdot}(1_{Q^n})[-n-1]$$

where $P^n, Q^n$ are projective (resp., injective) objects of $\mathcal{A}$.

**Lemma 13.10.** *For a complex $Z^{\boldsymbol{\cdot}} \in \mathsf{C}(\mathcal{A})$, the following hold.*

1. *$M^{\boldsymbol{\cdot}}$ is proper projective if and only if for every proper epimorphism $g : X^{\boldsymbol{\cdot}} \to Y^{\boldsymbol{\cdot}}$ in $\mathsf{C}(\mathcal{A})$, $\mathrm{Hom}_{\mathsf{C}(\mathcal{A})}(M^{\boldsymbol{\cdot}}, g)$ is surjective.*
2. *$M^{\boldsymbol{\cdot}}$ is proper injective if and only if for every proper monomorphism $f : X^{\boldsymbol{\cdot}} \to Y^{\boldsymbol{\cdot}}$ in $\mathsf{C}(\mathcal{A})$, $\mathrm{Hom}_{\mathsf{C}(\mathcal{A})}(f, M^{\boldsymbol{\cdot}})$ is surjective.*

*Proof.* 1. If $M^{\boldsymbol{\cdot}}$ is proper projective, then it suffices to check the case that $M^{\boldsymbol{\cdot}} = P[-n]$ or $\mathrm{M}^{\boldsymbol{\cdot}}(1_P)[-n]$, where $P$ is a projective object of $\mathcal{A}$, $n \in \mathbb{Z}$. For any proper epimorphism $g : X^{\boldsymbol{\cdot}} \to Y^{\boldsymbol{\cdot}}$, we have commutative diagrams

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{C}(\mathcal{A})}(P[-n], X^{\boldsymbol{\cdot}}) & \xrightarrow{\mathrm{Hom}_{\mathsf{C}(\mathcal{A})}(P[-n],g)} & \mathrm{Hom}_{\mathsf{C}(\mathcal{A})}(P[-n], Y^{\boldsymbol{\cdot}}) \\ \downarrow\wr & & \downarrow\wr \\ \mathrm{Hom}_{\mathcal{A}}(P, \mathrm{Z}^n(X^{\boldsymbol{\cdot}})) & \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(P,\mathrm{Z}^n(g))} & \mathrm{Hom}_{\mathcal{A}}(P, \mathrm{Z}^n(Y^{\boldsymbol{\cdot}})), \end{array}$$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{C}(\mathcal{A})}(\mathrm{M}^{\boldsymbol{\cdot}}(1_P)[-n], X^{\boldsymbol{\cdot}}) & \xrightarrow{\mathrm{Hom}_{\mathsf{C}(\mathcal{A})}(\mathrm{M}^{\boldsymbol{\cdot}}(1_P)[-n],g)} & \mathrm{Hom}_{\mathsf{C}(\mathcal{A})}(\mathrm{M}^{\boldsymbol{\cdot}}(1_P)[-n], Y^{\boldsymbol{\cdot}}) \\ \downarrow\wr & & \downarrow\wr \\ \mathrm{Hom}_{\mathcal{A}}(P, X^{n-1}) & \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(P,g^{n-1})} & \mathrm{Hom}_{\mathcal{A}}(P, Y^{n-1}). \end{array}$$

Since $P$ is projective, the bottom arrows of the above diagrams are surjective. Then the top arrows are also surjective. Conversely, let $M^{\boldsymbol{\cdot}}$ be a complex satisfying the surjective condition. For any epimorphism $g : X \to Y$ in $\mathcal{A}$, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{C}(\mathcal{A})}(M^{\boldsymbol{\cdot}}, X[-n]) & \xrightarrow{\mathrm{Hom}_{\mathsf{C}(\mathcal{A})}(M^{\boldsymbol{\cdot}},g[-n])} & \mathrm{Hom}_{\mathsf{C}(\mathcal{A})}(M^{\boldsymbol{\cdot}}, Y[-n]) \\ \downarrow\wr & & \downarrow\wr \\ \mathrm{Hom}_{\mathcal{A}}(\mathrm{C}^n(M^{\boldsymbol{\cdot}}), X) & \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(\mathrm{C}^n(M^{\boldsymbol{\cdot}}),g)} & \mathrm{Hom}_{\mathcal{A}}(\mathrm{C}^n(M^{\boldsymbol{\cdot}}), Y). \end{array}$$

Since $g[-n] : X[-n] \to Y[-n]$ is proper epic, the top arrow is surjective. Then $\mathrm{C}^n(M^{\boldsymbol{\cdot}})$ are projective objects of $\mathcal{A}$. It is easy to see that

$$M^{\boldsymbol{\cdot}} \cong \bigoplus\nolimits_{n\in\mathbb{Z}} \mathrm{M}^{\boldsymbol{\cdot}}(p^n)[-n-1],$$

where $p^n : \mathrm{C}^n(M^{\boldsymbol{\cdot}}) \to \mathrm{B}^n(M^{\boldsymbol{\cdot}})$ are the canonical epimorphisms. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{C}(\mathcal{A})}(\mathrm{M}^{\boldsymbol{\cdot}}(p^n), \mathrm{M}^{\boldsymbol{\cdot}}(1_X)) & \xrightarrow{\mathrm{Hom}_{\mathsf{C}(\mathcal{A})}(\mathrm{M}^{\boldsymbol{\cdot}}(p^n),\mathrm{M}^{\boldsymbol{\cdot}}(g))} & \mathrm{Hom}_{\mathsf{C}(\mathcal{A})}(\mathrm{M}^{\boldsymbol{\cdot}}(p^n), \mathrm{M}^{\boldsymbol{\cdot}}(1_X)) \\ \downarrow\wr & & \downarrow\wr \\ \mathrm{Hom}_{\mathcal{A}}(\mathrm{B}^n(M^{\boldsymbol{\cdot}}), X) & \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(\mathrm{B}^n(M^{\boldsymbol{\cdot}}),g)} & \mathrm{Hom}_{\mathcal{A}}(\mathrm{B}^n(M^{\boldsymbol{\cdot}}), Y). \end{array}$$

Since $\mathrm{M}^{\boldsymbol{\cdot}}(p^n)$ is proper projective, the top arrow is surjective, and then the bottom arrow is surjective. Therefore $\mathrm{B}^n(M^{\boldsymbol{\cdot}})$ is a projective object of $\mathcal{A}$. Hence $M^{\boldsymbol{\cdot}}$ is proper projective.

2. Similarly. □

**Lemma 13.11** (Proper Resolutions)**.** *Assume that $\mathcal{A}$ has enough projectives (resp. injectives). Given a complex $M^{\boldsymbol{\cdot}} \in \mathsf{C}(\mathcal{A})$, there are proper projective complexes* *(resp., proper injective complexes)* $X^{\cdot n}$ $(n \geq 0)$ *which has a proper projective resolution (resp., a proper injective resolution) in* $\mathsf{C}(\mathcal{A})$

$$\ldots \to X^{\cdot -1} \to X^{\cdot 0} \to M^{\cdot} \to O$$
$$(\textit{resp., } O \to M^{\cdot} \to X^{\cdot 0} \to X^{\cdot 1} \to \ldots).$$

*Proof.* For a complex $M^{\cdot} \in \mathsf{C}(\mathcal{A})$, we have exact sequences in $\mathsf{C}(\mathcal{A})$

$$O \to \mathrm{Z}^{\cdot}(M^{\cdot}) \to M^{\cdot} \to \mathrm{B}^{\cdot}(M^{\cdot})[1] \to O$$
$$O \to \mathrm{B}^{\cdot}(M^{\cdot}) \to \mathrm{Z}^{\cdot}(M^{\cdot}) \to \mathrm{H}^{\cdot}(M^{\cdot}) \to O.$$

For each $\mathrm{B}^n(M^{\cdot})$, there is a projective object $Q^n$ which has a epimorphism $Q^n \to \mathrm{B}^n(M^{\cdot})$. Then we have an epimorphism $\bigoplus_{n\in\mathbb{Z}} \mathrm{M}^{\cdot}(1_{Q^n})[-n] \to \mathrm{B}^{\cdot}(M^{\cdot})[1]$ and its lift $\bigoplus_{n\in\mathbb{Z}} \mathrm{M}^{\cdot}(1_{Q^n})[-n] \to M^{\cdot}$. For each $\mathrm{H}^n(M^{\cdot})$, there is a projective object $P^n$ which has a epimorphism $P^n \to \mathrm{H}^n(M^{\cdot})$. Then we have a morphism $\bigoplus_{n\in\mathbb{Z}} P^n[-n] \to \mathrm{B}^{\cdot}(M^{\cdot})$ and its lift $\bigoplus_{n\in\mathbb{Z}} P^n[-n] \to \mathrm{Z}^{\cdot}(M^{\cdot}) \to M^{\cdot}$. Then it is easy to see that $\bigoplus_{n\in\mathbb{Z}} P^n[-n] \oplus \bigoplus_{n\in\mathbb{Z}} \mathrm{M}^{\cdot}(1_{Q^n})[-n] \to M^{\cdot}$ is proper epimorphism. Then by induction we complete the proof. □

The above proper resolution $\ldots \to X^{\cdot -1} \to X^{\cdot 0}$ (resp., $X^{\cdot 0} \to X^{\cdot 1} \to \ldots$) is called a proper projective (resp., injective) resolution of $M^{\cdot}$ (they are often called Cartan-Eilenberg resolutions).

**Proposition 13.12.** *The following hold.*

1. *Assume that* $\mathcal{A}$*satisfies the condition Ab4 with enough projectives. Given a complex* $M^{\cdot} \in \mathsf{C}(\mathcal{A})$*, let* $\pi : P^{\cdot\cdot} \to M^{\cdot}$ *be a proper projective resolution. Then*
$$\operatorname{Tot} \pi : \operatorname{Tot} P^{\cdot\cdot} \to M^{\cdot}$$
*is a quasi-isomorphism in* $\mathsf{K}(\mathcal{A})$*, and* $\operatorname{Tot} P^{\cdot\cdot} \in \mathsf{K}^{\mathrm{s}}(\mathsf{Proj}\,\mathcal{A})$.
2. *Assume that* $\mathcal{A}$*satisfies the condition Ab4* with enough injectives. Given a complex* $M^{\cdot} \in \mathsf{C}(\mathcal{A})$*, let* $\mu : M^{\cdot} \to I^{\cdot\cdot}$ *be a proper injective resolution. Then*
$$\overset{\wedge}{\operatorname{Tot}}\mu : M^{\cdot} \to \overset{\wedge}{\operatorname{Tot}} I^{\cdot\cdot}$$
*is a quasi-isomorphism in* $\mathsf{K}(\mathcal{A})$*, and* $\overset{\wedge}{\operatorname{Tot}} I^{\cdot\cdot} \in \mathsf{K}^{\mathrm{s}}(\mathsf{Inj}\,\mathcal{A})$.

*Proof.* 1. We can consider $\pi : P^{\cdot\cdot} \to M^{\cdot}$ as $\pi : P^{\cdot\cdot} \to em^{\mathrm{I}} M^{\cdot}$ in $\mathsf{C}^2(\mathcal{A})$. Then we have a commutative diagram

$$\begin{array}{ccc} \sigma^{\mathrm{I}}_{\leq n} P^{\cdot\cdot} & \xrightarrow{\sigma^{\mathrm{I}}_{\leq n}\pi} & \sigma^{\mathrm{I}}_{\leq n} em^{\mathrm{I}} M^{\cdot} \\ {\scriptstyle \alpha_n}\downarrow & & \downarrow{\scriptstyle \beta_n} \\ \sigma^{\mathrm{I}}_{\leq n+1} P^{\cdot\cdot} & \xrightarrow{\sigma^{\mathrm{I}}_{\leq n}\pi} & \sigma^{\mathrm{I}}_{\leq n+1} em^{\mathrm{I}} M^{\cdot} \end{array}$$

where $\alpha_n$, $\beta_n$ are term-split monomorphisms. Therefore we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Tot} \sigma^{\mathrm{I}}_{\leq n} P^{\cdot\cdot} & \xrightarrow{\sigma^{\mathrm{I}}_{\leq n}\pi} & \sigma_{\leq n} M^{\cdot} \\ {\scriptstyle \operatorname{Tot}\alpha_n}\downarrow & & \downarrow{\scriptstyle \beta_n} \\ \operatorname{Tot} \sigma^{\mathrm{I}}_{\leq n+1} P^{\cdot\cdot} & \xrightarrow{\sigma^{\mathrm{I}}_{\leq n}\pi} & \sigma_{\leq n+1} M^{\cdot} \end{array}$$

where $\operatorname{Tot}\alpha_n$, $\beta_n$ are term-split monomorphisms, and all horizontal morphisms are quasi-isomorphisms, because $\operatorname{Tot}(\sigma^{\mathrm{I}}_{\leq n}P^{\cdot\cdot} \xrightarrow{\sigma^{\mathrm{I}}_{\leq n}\pi} \sigma^{\mathrm{I}}_{\leq n}em^{\mathrm{I}}M^{\cdot})$ is acyclic by Lemma 13.7. According to Proposition 11.7, we have isomorphisms in $\mathsf{D}(\mathcal{A})$

$$\begin{aligned}\operatorname{Tot}P^{\cdot\cdot} &= \varinjlim \operatorname{Tot}\sigma^{\mathrm{I}}_{\leq n}P^{\cdot\cdot}\\ &\cong \underset{\longrightarrow}{\operatorname{hlim}}\ \operatorname{Tot}\sigma^{\mathrm{I}}_{\leq n}P^{\cdot\cdot}\\ &\cong \underset{\longrightarrow}{\operatorname{hlim}}\ \sigma_{\leq n}M^{\cdot}\\ &\cong \varinjlim \sigma_{\leq n}M^{\cdot}\\ &= M^{\cdot}.\end{aligned}$$

On the other hand, since $\operatorname{Tot}\sigma^{\mathrm{I}}_{\leq n}P^{\cdot\cdot} \in \mathsf{K}^{-}(\mathsf{Proj}\,\mathcal{A})$, $\operatorname{Tot}P^{\cdot\cdot} \cong \underset{\longrightarrow}{\operatorname{hlim}}\ \operatorname{Tot}\sigma^{\mathrm{I}}_{\leq n}P^{\cdot\cdot} \in \mathsf{K}^{\mathrm{s}}(\mathsf{Proj}\,\mathcal{A})$.

2. Similarly □

**Definition 13.13.** For a morphism $u : X^{\cdot\cdot} \to Y^{\cdot\cdot}$ of double complexes, we define the complexes $\mathrm{M}^{\cdot\cdot}_{\mathrm{I}}(u)$, $\mathrm{M}^{\cdot\cdot}_{\mathrm{II}}(u)$ as follows.

$$\begin{aligned}\mathrm{M}^{\cdot\cdot}_{\mathrm{I}}(u)^{p,q} &= X^{p+1,q} \oplus Y^{p,q}\\ d^{p,q}_{\mathrm{I}\,\mathrm{M}^{\cdot\cdot}_{\mathrm{I}}(u)} &= \begin{bmatrix} -d^{p+1,q}_{\mathrm{I}\,X} & 0\\ u^{p+1,q} & d^{p,q}_{\mathrm{I}\,Y}\end{bmatrix}\\ d^{p,q}_{\mathrm{II}\,\mathrm{M}^{\cdot\cdot}_{\mathrm{I}}(u)} &= \begin{bmatrix} -d^{p+1,q}_{\mathrm{II}\,X} & 0\\ 0 & d^{p,q}_{\mathrm{II}\,Y}\end{bmatrix},\end{aligned}$$

$$\begin{aligned}\mathrm{M}^{\cdot\cdot}_{\mathrm{II}}(u)^{p,q} &= X^{p,q+1} \oplus Y^{p,q}\\ d^{p,q}_{\mathrm{I}\,\mathrm{M}^{\cdot\cdot}_{\mathrm{II}}(u)} &= \begin{bmatrix} -d^{p,q+1}_{\mathrm{I}\,X} & 0\\ 0 & d^{p,q}_{\mathrm{I}\,Y}\end{bmatrix}\\ d^{p,q}_{\mathrm{II}\,\mathrm{M}^{\cdot\cdot}_{\mathrm{II}}(u)} &= \begin{bmatrix} -d^{p,q+1}_{\mathrm{II}\,X} & 0\\ u^{p+1,q} & d^{p,q}_{\mathrm{II}\,Y}\end{bmatrix}.\end{aligned}$$

**Proposition 13.14.** *For a morphism $u : X^{\cdot\cdot} \to Y^{\cdot\cdot}$ of double complexes, the following hold.*

1. $\operatorname{Tot}\mathrm{M}^{\cdot\cdot}_{\mathrm{I}}(u) = \mathrm{M}^{\cdot}(\operatorname{Tot}u)$.
2. $\operatorname{Tot}\mathrm{M}^{\cdot\cdot}_{\mathrm{II}}(u) = \mathrm{M}^{\cdot}(\operatorname{Tot}u)$.

## 14. Derived Functors of Bi-∂-functors

**Definition 14.1** (Bi-$\partial$-functor)**.** For triangulated categories $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{D}$, *a bi-$\partial$-functor* $(F, \theta_1, \theta_2) : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$ is a bifunctor $F : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$ together with bifunctorial isomorphisms $\theta_1 : F(T_{\mathcal{C}_1}-, ?) \xrightarrow{\sim} T_{\mathcal{D}}F(-, ?)$, $\theta_2 : F(-, T_{\mathcal{C}_2}?) \xrightarrow{\sim} T_{\mathcal{D}}F(-, ?)$ such that

(a) For each object $X_1 \in \mathcal{C}_1$, $F(X_1, -) = (F(X_1, -), \theta_{2(X_1,-)}) : \mathcal{C}_2 \to \mathcal{D}$ is a $\partial$-functor.

(b) For each object $X_2 \in \mathcal{C}_2$, $F(-, X_2) = (F(-, X_2), \theta_{1(-,X_2)}) : \mathcal{C}_1 \to \mathcal{D}$ is a $\partial$-functor.

For bi-$\partial$-functors $(F, \theta_1, \theta_2), (G, \eta_1, \eta_2) : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$, a bifunctorial morphism $\phi : F \to G$ is called a *bi-$\partial$-functorial morphism* if $(T_{\mathcal{D}}\phi)\theta_1 = \eta_1\phi(T_{\mathcal{C}_1} \times \mathbf{1}_{\mathcal{C}_2})$, $(T_{\mathcal{D}}\phi)\theta_2 = \eta_2\phi(\mathbf{1}_{\mathcal{C}_1} \times T_{\mathcal{C}_2})$.

We denote by $\partial^2(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{D})$ the collection of all bi-$\partial$-functors from $\mathcal{C}_1 \times \mathcal{C}_2$ to $\mathcal{D}$, and denote by $\partial^2\operatorname{Mor}(F, G)$ the collection of all bi-$\partial$-functorial morphisms from $F$ to $G$.

**Proposition 14.2.** *Let $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$ be triangulated categories, $\mathcal{U}_i$ épaisse subcategories of $\mathcal{C}_i$, and $Q_i : \mathcal{U}_i \to \mathcal{C}_i/\mathcal{U}_i$ the canonical quotients $(i = 1, 2)$. Assume a bi-$\partial$-functor $F = (F, \theta_1, \theta_2) : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_3$ satisfies that*

(a) $F(\mathcal{U}_1, \mathcal{C}_2) = \{O\}$.
(b) $F(\mathcal{C}_1, \mathcal{U}_2) = \{O\}$.

*Then there exists a unique bi-$\partial$-functor $\overline{F} = (\overline{F}, \overline{\theta}_1, \overline{\theta}_2) : \mathcal{C}_1/\mathcal{U}_1 \times \mathcal{C}_2/\mathcal{U}_2 \to \mathcal{C}_3$ such that $F = \overline{F}(Q_1 \times Q_2)$ and $\theta_i = \overline{\theta}_i(Q_1 \times Q_2)$ $(i = 1, 2)$.*

$$\begin{array}{ccc} \mathcal{C}_1 \times \mathcal{C}_2 & & \\ {\scriptstyle Q_1\times Q_2}\downarrow & \overset{F}{\searrow} & \\ \mathcal{C}_1/\mathcal{U}_1 \times \mathcal{C}_2/\mathcal{U}_2 & \underset{\overline{F}}{\dashrightarrow} & \mathcal{C}_3 \end{array}$$

*Sketch.* We define the functor $\overline{F} : \mathcal{C}_1/\mathcal{U}_1 \times \mathcal{C}_2/\mathcal{U}_2 \to \mathcal{C}_3$ as follows. For $X_i \in \mathcal{C}_i$, let $\overline{F}(Q_1X_1, Q_2X_2) = F(X_1, X_2)$. For $[(f_i, s_i)] : X_i \to Y_i$ in $\mathcal{C}_i/\mathcal{U}_i$, let $\overline{F}([(f_1, s_1)], Q_2X_2) = F(s_1, X_2)^{-1}F(f_1, X_2)$, $\overline{F}(Q_1X_1, [(f_2, s_2)]) = F(X_1, s_2)^{-1}F(X_1, f_2)$. Let $T_i$, $\overline{T}_i$ are translations of $\mathcal{C}_i$, $\mathcal{C}_i/\mathcal{U}_i$, respectively $(i = 1, 2, 3)$. Then we define

$$\begin{array}{ccccc} \overline{F}(\overline{T}_1Q_1X_1, Q_2X_2) & = & \overline{F}(Q_1T_1X_1, Q_2X_2) & = & F(T_1X_1, X_2) \\ & {\scriptstyle \overline{\theta}_1(Q_1X_1,Q_2X_2)}\searrow & & & \downarrow{\scriptstyle \theta_1(X_1,X_2)} \\ & & T_3\overline{F}(Q_1X_1, Q_2X_2) & = & T_3F(X_1, X_2) \end{array}$$

$$\begin{array}{ccccc} \overline{F}(Q_1X_1, \overline{T}_2Q_2X_2) & = & \overline{F}(Q_1X_1, Q_2T_2X_2) & = & F(X_1, T_2X_2) \\ & {\scriptstyle \overline{\theta}_2(Q_1X_1,Q_2X_2)}\searrow & & & \downarrow{\scriptstyle \theta_2(X_1,X_2)} \\ & & T_3\overline{F}(Q_1X_1, Q_2X_2) & = & T_3F(X_1, X_2) \end{array}$$

Then it is not hard to see that $\overline{F}$ satisfies the assertions (left to the reader). □

**Proposition 14.3.** *Let $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$ be triangulated categories, $\mathcal{U}_i$ épaisse subcategories of $\mathcal{C}_i$, and $Q_i : \mathcal{U}_i \to \mathcal{C}_i/\mathcal{U}_i$ the canonical quotients $(i = 1, 2)$. For bi-$\partial$-functors $F = (F, \theta_1, \theta_2), G = (G, \eta_1, \eta_2) : \mathcal{C}_1/\mathcal{U}_1 \times \mathcal{C}_2/\mathcal{U}_2 \to \mathcal{C}_3$, we have a bijective correspondence*

$$\partial^2 \operatorname{Mor}(F, G) \xrightarrow{\sim} \partial^2 \operatorname{Mor}(F(Q_1 \times Q_2), G(Q_1 \times Q_2)), \quad (\zeta \mapsto \zeta(Q_1 \times Q_2)).$$

**Definition 14.4** (Right Derived Functor)**.** Let $\mathcal{A}_i$ be abelian categories, and let $\mathsf{K}^{*_i}(\mathcal{A}_i)$ be a quotientizing subcategory of $\mathsf{K}(\mathcal{A}_i)$ and $F : \mathsf{K}^{*_1}(\mathcal{A}_1) \times \mathsf{K}^{*_2}(\mathcal{A}_2) \to \mathsf{K}(\mathcal{A}_3)$ a bi-$\partial$-functor $(i = 1, 2, 3)$. The *right derived functor* of $F$ is a bi-$\partial$-functor

$$\boldsymbol{R}^{*_1,*_2}F : \mathsf{D}^{*_1}(\mathcal{A}_1) \times \mathsf{D}^{*_2}(\mathcal{A}_2) \to \mathsf{D}(\mathcal{A}_3)$$

together with a functorial morphism of bi-$\partial$-functors

$$\xi \in \partial^2 \operatorname{Mor}(Q_{\mathcal{A}_3}F, \boldsymbol{R}^{*_1,*_2}F(Q^{*_1}_{\mathcal{A}_1} \times Q^{*_2}_{\mathcal{A}_2}))$$

with the following property:
For $G \in \partial^2(\mathsf{D}^{*_1}(\mathcal{A}_1) \times \mathsf{D}^{*_2}(\mathcal{A}_2), \mathsf{D}(\mathcal{A}_3))$ and $\zeta \in \partial^2 \operatorname{Mor}(Q_{\mathcal{A}_3}F, G(Q^{*_1}_{\mathcal{A}_1} \times Q^{*_2}_{\mathcal{A}_2}))$, there exists a unique morphism $\eta \in \partial^2 \operatorname{Mor}(\boldsymbol{R}^{*_1,*_2}F, G)$ such that

$$\zeta = (\eta(Q^{*_1}_{\mathcal{A}_1} \times Q^{*_2}_{\mathcal{A}_2}))\xi.$$

In other words,

$$\partial^2 \operatorname{Mor}(Q_{\mathcal{A}_3}F, -(Q^{*_1}_{\mathcal{A}_1} \times Q^{*_2}_{\mathcal{A}_2})) \cong \partial^2 \operatorname{Mor}(\boldsymbol{R}^{*_1,*_2}F, -)$$

as functors from $\partial^2(\mathsf{D}^{*_1}(\mathcal{A}_1) \times \mathsf{D}^{*_2}(\mathcal{A}_2), \mathsf{D}(\mathcal{A}_3))$ to $\mathfrak{Set}$ (See Lemma 1.8).

**Theorem 14.5** (Existence Theorem). *Let $\mathcal{A}_i$ be abelian categories ($i = 1, 2, 3$), and let $\mathsf{K}^{*_i}(\mathcal{A}_i)$ be a quotientizing subcategory of $\mathsf{K}(\mathcal{A}_i)$ and $F : \mathsf{K}^{*_1}(\mathcal{A}_1) \times \mathsf{K}^{*_2}(\mathcal{A}_2) \to \mathsf{K}(\mathcal{A}_3)$ a bi-$\partial$-functor. Assume there exist triangulated full subcategories $\mathcal{L}_i$ of $\mathsf{K}^{*_i}(\mathcal{A}_i)$ ($i = 1, 2$) such that*

(a) *for any $X^{\cdot}_i \in \mathsf{K}^{*_i}(\mathcal{A}_i)$ there is a quasi-isomorphism $X^{\cdot}_i \to I^{\cdot}_i$ with $I^{\cdot}_i \in \mathcal{L}_i$,*
(b) *$Q_{\mathcal{A}_3}F(\mathcal{L}^{\phi}_1, \mathcal{L}_2) = \{O\}$,*
(c) *$Q_{\mathcal{A}_3}F(\mathcal{L}_1, \mathcal{L}^{\phi}_2) = \{O\}$,*

*where $\mathcal{L}^{\phi}_i = \mathsf{K}^{\phi}(\mathcal{A}_i) \cap \mathcal{L}_i$ ($i = 1, 2$). Then there exists the right derived functor $(\boldsymbol{R}^{*_1,*_2}F, \xi)$ such that $\xi_{(I^{\cdot}_1, I^{\cdot}_2)} : Q_{\mathcal{A}_3}F(I^{\cdot}_1, I^{\cdot}_2) \to \boldsymbol{R}^{*}F(I^{\cdot}_1, I^{\cdot}_2)$ is a quasi-isomorphism for $(I^{\cdot}_1, I^{\cdot}_2) \in \mathcal{L}_1 \times \mathcal{L}_2$.*

*Proof.* Let $Q_i : \mathsf{K}^{*_i}(\mathcal{A}_i) \to \mathsf{D}^{*_i}(\mathcal{A}_i)$ be the canonical quotients, and let $E_i : \mathcal{L}_i \to \mathsf{K}^{*_i}(\mathcal{A}_i)$ be the embedding functors, then by the assumption 1 and Proposition 7.16 the canonical functor $\overline{E}_i : \mathcal{L}_i/\mathcal{L}^{\phi}_i \to \mathsf{D}^{*_i}(\mathcal{A}_i)$ are equivalences ($i = 1, 2$). Let $J_i : \mathsf{D}^{*_i}(\mathcal{A}_i) \to \mathcal{L}_i/\mathcal{L}^{\phi}_i$ be quasi-inverses of $\overline{E}_i$ ($i = 1, 2$). By the assumption 2, 3 and Proposition 14.2 there is a bi-$\partial$-functor

$$\overline{F} : \mathcal{L}_1/\mathcal{L}^{\phi}_1 \times \mathcal{L}_2/\mathcal{L}^{\phi}_2 \to \mathsf{D}(\mathcal{A}_3)$$

such that $Q_3F(E_1 \times E_2) = \overline{F}(Q'_1 \times Q'_2)$, where $Q'_i : \mathcal{L}_i \to \mathcal{L}_i/\mathcal{L}^{\phi}_i$ are the canonical quotients. Put $\boldsymbol{R}^{*_1,*_2}F = \overline{F}(J_1 \times J_2)$. Since $(Q_1E_1 \times Q_2E_2) = (\overline{E}_1Q'_1 \times \overline{E}_2Q'_2)$, we have

$$\begin{aligned}
&\partial^2 \operatorname{Mor}(Q_3F(E_1 \times E_2), G(Q_1E_1 \times Q_2E_2)) \\
&\cong \partial^2 \operatorname{Mor}(\overline{F}(Q'_1 \times Q'_2), G(\overline{E}_1Q'_1 \times \overline{E}_2Q'_2)) \\
&\cong \partial^2 \operatorname{Mor}(\overline{F}(J_1E_1 \times J_2E_2), G(\overline{E}_1 \times \overline{E}_2)) \\
&\cong \partial^2 \operatorname{Mor}(\overline{F}(J_1 \times J_2), G) \\
&= \partial^2 \operatorname{Mor}(\boldsymbol{R}^{*_1,*_2}F, G)
\end{aligned}$$

It remains to show that

$$\begin{aligned}
\partial^2 \operatorname{Mor}(Q_3F, G(Q_1 \times Q_2)) &\xrightarrow{\sim} \partial^2 \operatorname{Mor}(Q_3F(E_1 \times E_2), G(Q_1E_1 \times Q_2E_2)), \\
&(\phi \mapsto \phi(E_1 \times E_2)).
\end{aligned}$$

Let $\phi \in \partial^2 \operatorname{Mor}(Q_3F, G(Q_1 \times Q_2))$ with $\phi(E_1 \times E_2) = 0$. For any $X^{\cdot}_i \in \mathsf{K}^{*_i}(\mathcal{A}_i)$ there exists $I^{\cdot}_i \in \mathcal{L}_i$ which has a quasi-isomorphism $s_i : X^{\cdot}_i \to I^{\cdot}_i$ ($i = 1, 2$). Then

$$\begin{aligned}
\phi_{(X^{\cdot}_1, X^{\cdot}_2)} &= G(Q_1s_1, Q_2s_2)^{-1}\phi_{(I^{\cdot}_1, I^{\cdot}_2)}Q_3F(s_1, s_2) \\
&= 0,
\end{aligned}$$

and hence $\phi = 0$. Given $\psi \in \partial^2 \operatorname{Mor}(Q_3F(E_1 \times E_2), G(Q_1E_1 \times Q_2E_2))$, for any $X^{\cdot}_i \in \mathsf{K}^{*_i}(\mathcal{A}_i)$, let

$$\phi_{(X_1, X_2)} = (G(Q_1s_1, Q_2s_2))^{-1}\psi_{(I^{\cdot}_1, I^{\cdot}_2)}Q_3F(s_1, s_2)$$

for some quasi-isomorphism $s_i : X^{\cdot}_i \to I^{\cdot}_i$, with $I^{\cdot}_i \in \mathcal{L}_i$ ($i = 1, 2$). For another quasi-isomorphism $s'_i : X^{\cdot}_i \to I'^{\cdot}_i$, by the assumptions 1, we have commutative diagrams

$$\begin{array}{ccc} X_i^{\cdot} & \xrightarrow{s_i} & I_i^{\cdot} \\ {\scriptstyle s'_i}\downarrow & & \downarrow{\scriptstyle t'_i} \\ I'^{\cdot}_i & \xrightarrow{t_i} & I''^{\cdot}_i \end{array}$$

where all morphisms are quasi-isomorphisms and $I''^{\cdot}_i \in \mathcal{L}_i$ $(i = 1, 2)$. Then we have

$$\begin{aligned} &(G(Q_1 s_1, Q_2 s_2))^{-1}\psi_{(I_1^{\cdot}, I_2^{\cdot})} Q_3 F(s_1, s_2) \\ &= (G(Q_1 t'_1 s_1, Q_2 t'_2 s_2))^{-1}\psi_{(I''^{\cdot}_1, I''^{\cdot}_2)} Q_3 F(t'_1 s_1, t'_2 s_2) \\ &= (G(Q_1 t_1 s'_1, Q_2 t_2 s'_2))^{-1}\psi_{(I''^{\cdot}_1, I''^{\cdot}_2)} Q_3 F(t_1 s'_1, t_2 s'_2) \\ &= (G(Q_1 s'_1, Q_2 s'_2))^{-1}\psi_{(I'^{\cdot}_1, I'^{\cdot}_2)} Q_3 F(s'_1, s'_2) \end{aligned}$$

It is not hard to see that $\phi \in \partial^2 \operatorname{Mor}(Q_3 F, G(Q_1 \times Q_2))$. The last assertion is easy to check. □

**Proposition 14.6.** *Let $\mathcal{A}_i$ be abelian categories ($i = 1, 2, 3$), and let $\mathsf{K}^{*_i}(\mathcal{A}_i)$ be a quotientizing subcategory of $\mathsf{K}(\mathcal{A}_i)$ and $F : \mathsf{K}^{*_1}(\mathcal{A}_1) \times \mathsf{K}^{*_2}(\mathcal{A}_2) \to \mathsf{K}(\mathcal{A}_3)$ a bi-$\partial$-functor. Assume there exists a triangulated full subcategories $\mathcal{L}_i$ of $\mathsf{K}^{*_i}(\mathcal{A}_i)$ ($i = 1, 2$) such that*

(a) *for any $X_i^{\cdot} \in \mathsf{K}^{*_i}(\mathcal{A}_i)$ there is a quasi-isomorphism $X_i^{\cdot} \to I_i^{\cdot}$ with $I_i^{\cdot} \in \mathcal{L}_i$,*
(b) *$Q_{\mathcal{A}_3} F(\mathcal{L}_1^{\phi}, \mathsf{K}^{*_2}(\mathcal{A}_2)) = \{O\}$,*
(c) *$Q_{\mathcal{A}_3} F(\mathcal{L}_1, \mathcal{L}_2^{\phi}) = \{O\}$,*

*where $\mathcal{L}_i^{\phi} = \mathsf{K}^{\phi}(\mathcal{A}_i) \cap \mathcal{L}_i$ ($i = 1, 2$). Then we have*

1. *There is a bi-$\partial$-functor $\boldsymbol{R}_{\mathrm{I}}^{*_1,*_2} F : \mathsf{D}^{*_1}(\mathcal{A}_1) \times \mathsf{K}^{*_2}(\mathcal{A}_2) \to \mathsf{D}(\mathcal{A}_3)$ such that*
$$\partial^2 \operatorname{Mor}(Q_{\mathcal{A}_3} F, -(Q_{\mathcal{A}_1}^{*_1} \times \mathbf{1}_{\mathsf{K}^{*_2}(\mathcal{A}_2)})) \cong \partial^2 \operatorname{Mor}(\boldsymbol{R}_{\mathrm{I}}^{*_1,*_2} F, -),$$
*and $\boldsymbol{R}_{\mathrm{I}}^{*_1,*_2} F(-, X_2)$ is the right derived functor of $F(-, X_2)$ for any $X_2 \in \mathsf{K}^{*_2}(\mathcal{A}_2)$.*
2. *There is a bi-$\partial$-functor $\boldsymbol{R}_{\mathrm{II}}^{*_1,*_2} \boldsymbol{R}_{\mathrm{I}}^{*_1,*_2} F : \mathsf{D}^{*_1}(\mathcal{A}_1) \times \mathsf{D}^{*_2}(\mathcal{A}_2) \to \mathsf{D}(\mathcal{A}_3)$ such that*
$$\partial^2 \operatorname{Mor}(\boldsymbol{R}_{\mathrm{I}}^{*_1,*_2} F, -(\mathbf{1}_{\mathsf{D}^{*_1}(\mathcal{A}_1)} \times Q_{\mathcal{A}_2}^{*_2})) \cong \partial^2 \operatorname{Mor}(\boldsymbol{R}_{\mathrm{II}}^{*_1,*_2} \boldsymbol{R}_{\mathrm{I}}^{*_1,*_2} F, -),$$
*and*
$$\partial \operatorname{Mor}(\boldsymbol{R}_{\mathrm{I}}^{*_1,*_2} F(X_1, ?), -Q_{\mathcal{A}_2}^{*_2}) \cong \partial \operatorname{Mor}(\boldsymbol{R}_{\mathrm{II}}^{*_1,*_2} \boldsymbol{R}_{\mathrm{I}}^{*_1,*_2} F(X_1, ?), -)$$
*for any $X_1 \in \mathsf{K}^{*_1}(\mathcal{A}_1)$.*
3. *We have an isomorphism*
$$\partial^2 \operatorname{Mor}(Q_{\mathcal{A}_3} F, -(Q_{\mathcal{A}_1}^{*_1} \times Q_{\mathcal{A}_2}^{*_2})) \cong \partial^2 \operatorname{Mor}(\boldsymbol{R}_{\mathrm{II}}^{*_1,*_2} \boldsymbol{R}_{\mathrm{I}}^{*_1,*_2} F, -).$$

*In particular, $\boldsymbol{R}_{\mathrm{II}}^{*_1,*_2} \boldsymbol{R}_{\mathrm{I}}^{*_1,*_2} F \cong \boldsymbol{R}^{*_1,*_2} F$.*

*Proof.* According to the construction of the right derived functor of a bi-$\partial$-functor in the proof of Theorems 14.5, 12.5, it is easy (left to the reader). □

**Corollary 14.7.** *Let $\mathcal{A}_i$ be abelian categories ($i = 1, 2, 3$), and let $\mathsf{K}^{*_i}(\mathcal{A}_i)$ be a quotientizing subcategory of $\mathsf{K}(\mathcal{A}_i)$ and $F : \mathsf{K}^{*_1}(\mathcal{A}_1) \times \mathsf{K}^{*_2}(\mathcal{A}_2) \to \mathsf{K}(\mathcal{A}_3)$ a bi-$\partial$-functor. Assume there exists a triangulated full subcategories $\mathcal{L}_i$ of $\mathsf{K}^{*_i}(\mathcal{A}_i)$ ($i = 1, 2$) such that*

(a) *for any $X_i^{\cdot} \in \mathsf{K}^{*_i}(\mathcal{A}_i)$ there is a quasi-isomorphism $X_i^{\cdot} \to I_i^{\cdot}$ with $I_i^{\cdot} \in \mathcal{L}_i$,*

(b) $Q_{\mathcal{A}_3}F(\mathcal{L}_1^{\phi}, \mathsf{K}^{*_2}(\mathcal{A}_2)) = \{O\}$,
(c) $Q_{\mathcal{A}_3}F(\mathsf{K}^{*_1}(\mathcal{A}_1), \mathcal{L}_2^{\phi}) = \{O\}$,

*where* $\mathcal{L}_i^{\phi} = \mathsf{K}^{\phi}(\mathcal{A}_i) \cap \mathcal{L}_i$ $(i = 1, 2)$*. Then we have*

1. *There is a bi-$\partial$-functor* $\boldsymbol{R}_{\mathrm{I}}^{*_1,*_2}F : \mathsf{D}^{*_1}(\mathcal{A}_1) \times \mathsf{K}^{*_2}(\mathcal{A}_2) \to \mathsf{D}(\mathcal{A}_3)$ *such that*
$$\partial^2 \operatorname{Mor}(Q_{\mathcal{A}_3}F, -(Q_{\mathcal{A}_1}^{*_1} \times \mathbf{1}_{\mathsf{K}^{*_2}(\mathcal{A}_2)})) \cong \partial^2 \operatorname{Mor}(\boldsymbol{R}_{\mathrm{I}}^{*_1,*_2}F, -),$$
*and* $\boldsymbol{R}_{\mathrm{I}}^{*_1,*_2}F(-, X_2)$ *is the right derived functor of* $F(-, X_2)$ *for any* $X_2 \in \mathsf{K}^{*_2}(\mathcal{A}_2)$*.*
2. *There is a bi-$\partial$-functor* $\boldsymbol{R}_{\mathrm{II}}^{*_1,*_2}F : \mathsf{K}^{*_1}(\mathcal{A}_1) \times \mathsf{D}^{*_2}(\mathcal{A}_2) \to \mathsf{D}(\mathcal{A}_3)$ *such that*
$$\partial^2 \operatorname{Mor}(Q_{\mathcal{A}_3}F, -(\mathbf{1}_{\mathsf{K}^{*_1}(\mathcal{A}_1)} \times Q_{\mathcal{A}_2}^{*_2})) \cong \partial^2 \operatorname{Mor}(\boldsymbol{R}_{\mathrm{II}}^{*_1,*_2}F, -),$$
*and* $\boldsymbol{R}_{\mathrm{II}}^{*_1,*_2}F(X_1, -)$ *is the right derived functor of* $F(X_1, -)$ *for any* $X_1 \in \mathsf{K}^{*_1}(\mathcal{A}_1)$*.*
3. *We have an isomorphism*
$$\boldsymbol{R}^{*_1,*_2}F \cong \boldsymbol{R}_{\mathrm{II}}^{*_1,*_2}\boldsymbol{R}_{\mathrm{I}}^{*_1,*_2}F \cong \boldsymbol{R}_{\mathrm{I}}^{*_1,*_2}\boldsymbol{R}_{\mathrm{II}}^{*_1,*_2}F.$$

**Definition 14.8** ($\operatorname{Hom}^{\bullet}_{\mathcal{A}}$)**.** For a complexes $X^{\bullet}, Y^{\bullet} \in \mathsf{C}(\mathcal{A})$, we define the double complex $\operatorname{Hom}^{\bullet\bullet}_{\mathcal{A}}(X^{\bullet}, Y^{\bullet})$ by
$$\begin{aligned} \operatorname{Hom}^{p,q}_{\mathcal{A}}(X^{\bullet}, Y^{\bullet}) &= \operatorname{Hom}_{\mathcal{A}}(X^{-p}, Y^{q}) \\ d^{p,q}_{\mathrm{I}\ \operatorname{Hom}^{\bullet\bullet}_{\mathcal{A}}(X^{\bullet}, Y^{\bullet})} &= \operatorname{Hom}^{p,q}_{\mathcal{A}}(d^{-p-1}_{X^{\bullet}}, Y^{q}) \\ d^{p,q}_{\mathrm{II}\ \operatorname{Hom}^{\bullet\bullet}_{\mathcal{A}}(X^{\bullet}, Y^{\bullet})} &= (-1)^{p+q+1} \operatorname{Hom}^{p,q}_{\mathcal{A}}(X^{-p}, d^{q}_{Y^{\bullet}}), \end{aligned}$$
and define the complex $\operatorname{Hom}^{\bullet}_{\mathcal{A}}(X^{\bullet}, Y^{\bullet})$ by
$$\overset{\wedge}{\operatorname{Tot}} \operatorname{Hom}^{\bullet\bullet}_{\mathcal{A}}(X^{\bullet}, Y^{\bullet}).$$
Then it is easy to see that
$$\operatorname{Hom}^{\bullet}_{\mathcal{A}} : \mathsf{C}(\mathcal{A})^{\mathrm{op}} \times \mathsf{C}(\mathcal{A}) \to \mathsf{C}(\mathfrak{Ab})$$
is a bifunctor.

**Lemma 14.9.** *For complexes* $X^{\bullet}, Y^{\bullet} \in \mathsf{C}(\mathcal{A})$*, we have an isomorphism*
$$\mathrm{H}^n \operatorname{Hom}^{\bullet}_{\mathcal{A}}(X^{\bullet}, Y^{\bullet}) \cong \operatorname{Hom}_{\mathsf{K}(\mathcal{A})}(X^{\bullet}, T^n Y^{\bullet}).$$

*Proof.* By the definition, for $(u^{p,q})_{p+q=r} \in \operatorname{Hom}^{r}_{\mathcal{A}}(X^{\bullet}, Y^{\bullet})$ we have
$$\begin{aligned} &d^{r}_{\operatorname{Hom}^{\bullet}_{\mathcal{A}}(X,Y)}((u^{p,q})_{p+q=r}) \\ &= (u^{p-1,q}d^{-p}_{X} + (-1)^{p+q+1}d^{q}_{Y}u^{p,q-1})_{p+q=r} \in \operatorname{Hom}^{r+1}_{\mathcal{A}}(X^{\bullet}, Y^{\bullet}). \end{aligned}$$
Put $u = f, i = -p, r = n$, then $f^{-i,i+n} : X^{i} \to Y^{i+n}$ for all $i$ and we have
$$\begin{aligned} &d^{n}_{\operatorname{Hom}^{\bullet}_{\mathcal{A}}(X,Y)}((f^{-i,i+n})_{i\in\mathbb{Z}}) \\ &= (f^{-i-1,i+1+n}d^{i}_{X} - (-1)^{n}d^{i+n}_{Y}f^{-i,i+n})_{i\in\mathbb{Z}}. \end{aligned}$$
Then it is easy to see that $\operatorname{Ker} d^{n}_{\operatorname{Hom}^{\bullet}_{\mathcal{A}}(X,Y)} = \operatorname{Hom}_{\mathsf{C}(\mathcal{A})}(X^{\bullet}, T^n Y^{\bullet})$. Put $u = h, i = -p, r = n-1$, then $h^{-i,i+n-1} : X^{i} \to Y^{i+n-1}$ for all $i$ and we have
$$\begin{aligned} &d^{n-1}_{\operatorname{Hom}^{\bullet}_{\mathcal{A}}(X,Y)}((h^{-i,i+n-1})_{i\in\mathbb{Z}}) \\ &= (h^{-i-1,i+n}d^{i}_{X} + (-1)^{n}d^{i+n}_{Y}h^{-i,i+n-1})_{i\in\mathbb{Z}}. \end{aligned}$$
Then this means $\operatorname{Im} d^{n-1}_{\operatorname{Hom}^{\bullet}_{\mathcal{A}}(X,Y)} = \operatorname{Htp}_{\mathsf{C}(\mathcal{A})}(X^{\bullet}, T^n Y^{\bullet})$. □

**Lemma 14.10.** *For a complexes* $X^{\bullet}, Y^{\bullet} \in \mathsf{C}(\mathcal{A})$*, the following hold.*

1. $d_{\mathrm{Hom}^{\bullet}_{\mathcal{A}}(X^{\bullet},TY^{\bullet})} = -Td_{\mathrm{Hom}^{\bullet}_{\mathcal{A}}(X^{\bullet},Y^{\bullet})}$
2. $d_{\mathrm{Hom}^{\bullet}_{\mathcal{A}}(T^{-1}X^{\bullet},Y^{\bullet})} = Td_{\mathrm{Hom}^{\bullet}_{\mathcal{A}}(X^{\bullet},Y^{\bullet})}$
3. *Define* $\theta_1^{p,q} : \mathrm{Hom}^{p,q}_{\mathcal{A}}(T^{-1}X^{\bullet}, Y^{\bullet}) \to \mathrm{Hom}^{p-1,q}_{\mathcal{A}}(X^{\bullet}, Y^{\bullet})$ *by the identities, then we have an isomorphism*
$$\theta_1 : \mathrm{Hom}^{\bullet}_{\mathcal{A}} \circ (T^{-1} \times \mathbf{1}_{\mathsf{C}(\mathcal{A})}) \xrightarrow{\sim} T \circ \mathrm{Hom}^{\bullet}_{\mathcal{A}} .$$
4. *Define* $\theta_2^{p,q} : \mathrm{Hom}^{p,q}_{\mathcal{A}}(X^{\bullet}, TY^{\bullet}) \to \mathrm{Hom}^{p,q+1}_{\mathcal{A}}(X^{\bullet}, Y^{\bullet})$ *by* $(-1)^{p+q}$*, then we have an isomorphism*
$$\theta_2 : \mathrm{Hom}^{\bullet}_{\mathcal{A}} \circ (\mathbf{1}_{\mathsf{C}(\mathcal{A})} \times T) \xrightarrow{\sim} T \circ \mathrm{Hom}^{\bullet}_{\mathcal{A}} .$$

**Lemma 14.11.** *For a morphism* $u : X^{\bullet} \to Y^{\bullet}$ *in* $\mathsf{C}(\mathcal{A})$ *and* $N^{\bullet} \in \mathsf{C}(\mathcal{A})$*, the following hold.*

1. $\mathrm{Hom}^{\bullet}_{\mathcal{A}}(N^{\bullet}, \mathrm{M}^{\bullet}(u)) \cong \mathrm{M}^{\bullet}(\mathrm{Hom}^{\bullet}_{\mathcal{A}}(N^{\bullet}, u))$.
2. $\mathrm{Hom}^{\bullet}_{\mathcal{A}}(\mathrm{M}^{\bullet}(u), N^{\bullet}) \cong T^{-1}\,\mathrm{M}^{\bullet}(\mathrm{Hom}^{\bullet}_{\mathcal{A}}(u, N^{\bullet}))$.

*Proof.* 1. The double complex $\mathrm{Hom}^{\bullet\bullet}_{\mathcal{A}}(N^{\bullet}, \mathrm{M}^{\bullet}(u))$ has the following form
$$\begin{aligned}
\mathrm{Hom}^{p,q}_{\mathcal{A}}(N^{\bullet}, \mathrm{M}^{\bullet}(u)) &= \mathrm{Hom}_{\mathcal{A}}(M^{-p}, X^{q+1} \oplus Y^{q}) \\
d^{p,q}_{\mathrm{I}\,\mathrm{Hom}^{\bullet\bullet}_{\mathcal{A}}(N^{\bullet},\mathrm{M}^{\bullet}(u))} &= \begin{bmatrix} \mathrm{Hom}(d_M^{-(p+1)}, X) & 0 \\ 0 & \mathrm{Hom}(d_M^{-(p+1)}, Y) \end{bmatrix} \\
d^{p,q}_{\mathrm{II}\,\mathrm{Hom}^{\bullet\bullet}_{\mathcal{A}}(N^{\bullet},\mathrm{M}^{\bullet}(u))} &= \begin{bmatrix} (-1)^{p+q+2}\,\mathrm{Hom}(M, d_X^{q+1}) & 0 \\ (-1)^{p+q+1}\,\mathrm{Hom}(M, u^{q}) & (-1)^{p+q+1}\,\mathrm{Hom}(M, d_Y^{q}) \end{bmatrix}
\end{aligned}$$
On the other hand, the double complex $\mathrm{M}^{\bullet\bullet}_{\mathrm{II}}(\mathrm{Hom}^{\bullet\bullet}_{\mathcal{A}}(N^{\bullet}, u))$ has the following form
$$\begin{aligned}
\mathrm{M}^{p,q}_{\mathrm{II}}(\mathrm{Hom}^{\bullet\bullet}_{\mathcal{A}}(N^{\bullet}, u)) &= \mathrm{Hom}_{\mathcal{A}}(M^{-p}, X^{q+1} \oplus Y^{q}) \\
d^{p,q}_{\mathrm{I}\,\mathrm{M}^{\bullet\bullet}_{\mathrm{II}}(\mathrm{Hom}^{\bullet\bullet}_{\mathcal{A}}(N^{\bullet},u))} &= \begin{bmatrix} -\mathrm{Hom}(d_M^{-(p+1)}, X) & 0 \\ 0 & \mathrm{Hom}(d_M^{-(p+1)}, Y) \end{bmatrix} \\
d^{p,q}_{\mathrm{II}\,\mathrm{M}^{\bullet\bullet}_{\mathrm{II}}(\mathrm{Hom}^{\bullet\bullet}_{\mathcal{A}}(N^{\bullet},u))} &= \begin{bmatrix} (-1)^{p+q+1}\,\mathrm{Hom}(M, d_X^{q+1}) & 0 \\ \mathrm{Hom}(M, u^{q}) & (-1)^{p+q+1}\,\mathrm{Hom}(M, d_Y^{q}) \end{bmatrix}
\end{aligned}$$
Then it is easy to see that morphisms $\left[\begin{smallmatrix} (-1)^{p+q} & 0 \\ 0 & 1 \end{smallmatrix}\right] : \mathrm{M}^{p,q}_{\mathrm{II}}(\mathrm{Hom}^{\bullet\bullet}_{\mathcal{A}}(N^{\bullet}, u)) \to \mathrm{Hom}^{p,q}_{\mathcal{A}}(N^{\bullet}, \mathrm{M}^{\bullet}(u))$ induce an isomorphism between them in $\mathsf{C}^2(\mathcal{A})$. By Proposition 13.14, we get the statement.

2. The double complex $\mathrm{Hom}^{\bullet\bullet}_{\mathcal{A}}(\mathrm{M}^{\bullet}(u), N^{\bullet})$ has the following form
$$\begin{aligned}
\mathrm{Hom}^{p,q}_{\mathcal{A}}(\mathrm{M}^{\bullet}(u), N^{\bullet}) &= \mathrm{Hom}_{\mathcal{A}}(X^{-p+1} \oplus Y^{-p}, M^{q}) \\
d^{p,q}_{\mathrm{I}\,\mathrm{Hom}^{\bullet\bullet}_{\mathcal{A}}(\mathrm{M}^{\bullet}(u),N^{\bullet})} &= \begin{bmatrix} -\mathrm{Hom}(d_X^{-p}, M) & 0 \\ \mathrm{Hom}(u^{p}, M) & \mathrm{Hom}(d_Y^{-p-1}, M) \end{bmatrix} \\
d^{p,q}_{\mathrm{II}\,\mathrm{Hom}^{\bullet\bullet}_{\mathcal{A}}(\mathrm{M}^{\bullet}(u),N^{\bullet})} &= \begin{bmatrix} (-1)^{p+q+1}\,\mathrm{Hom}(X, d_M^{q}) & 0 \\ 0 & (-1)^{p+q+1}\,\mathrm{Hom}(X, d_M^{q}) \end{bmatrix}
\end{aligned}$$
On the other hand, the double complex $T_{\mathrm{I}}^{-1}\,\mathrm{M}^{\bullet\bullet}_{\mathrm{I}}(\mathrm{Hom}^{\bullet\bullet}_{\mathcal{A}}(u, N^{\bullet}))$ has the following form
$$\begin{aligned}
(T_{\mathrm{I}}^{-1}\,\mathrm{M}^{\bullet\bullet}_{\mathrm{I}}(\mathrm{Hom}^{\bullet\bullet}_{\mathcal{A}}(u, N^{\bullet})))^{p,q} &= \mathrm{Hom}_{\mathcal{A}}(X^{-p+1} \oplus Y^{-p}, M^{q}) \\
d^{p,q}_{\mathrm{I}\,T_{\mathrm{I}}^{-1}\,\mathrm{M}^{\bullet\bullet}_{\mathrm{I}}(\mathrm{Hom}^{\bullet\bullet}_{\mathcal{A}}(u,N^{\bullet}))} &= \begin{bmatrix} \mathrm{Hom}(d_X^{-p}, M) & 0 \\ -\mathrm{Hom}(u^{p}, M) & -\mathrm{Hom}(d_Y^{-p-1}, M) \end{bmatrix} \\
d^{p,q}_{\mathrm{II}\,T_{\mathrm{I}}^{-1}\,\mathrm{M}^{\bullet\bullet}_{\mathrm{I}}(\mathrm{Hom}^{\bullet\bullet}_{\mathcal{A}}(u,N^{\bullet}))} &= \begin{bmatrix} (-1)^{p+q+1}\,\mathrm{Hom}(X, d_M^{q}) & 0 \\ 0 & (-1)^{p+q+1}\,\mathrm{Hom}(X, d_M^{q}) \end{bmatrix}
\end{aligned}$$
Then it is easy to see that morphisms $\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] : (T_{\mathrm{I}}^{-1}\,\mathrm{M}^{\bullet\bullet}_{\mathrm{I}}(\mathrm{Hom}^{\bullet\bullet}_{\mathcal{A}}(u, N^{\bullet})))^{p,q} \to \mathrm{Hom}^{p,q}_{\mathcal{A}}(\mathrm{M}^{\bullet}(u), N^{\bullet})$ induce an isomorphism between them in $\mathsf{C}^2(\mathcal{A})$. By Lemma 13.6 and Proposition 13.14, we get the statement. ☐

**Proposition 14.12.** *The bi-functor* $\mathrm{Hom}^{\bullet}_{\mathcal{A}} : \mathsf{C}(\mathcal{A})^{\mathrm{op}} \times \mathsf{C}(\mathcal{A}) \to \mathsf{C}(\mathfrak{Ab})$ *induces the bi-∂-functor*

$$\mathrm{Hom}^{\bullet}_{\mathcal{A}} : \mathsf{K}(\mathcal{A})^{\mathrm{op}} \times \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathfrak{Ab}).$$

*Proof.* By Lemmas 14.10, 14.11 and Corollary 6.16, it is easy. □

**Proposition 14.13.** *Let* $F : \mathcal{A} \to \mathcal{A}'$, $G : \mathcal{A}' \to \mathcal{A}$ *be additive functors between abelian categories such that* $G \dashv F$. *Then we have a functorial isomorphism*

$$\mathrm{Hom}^{\bullet}_{\mathcal{A}}(GX^{\bullet}, Y^{\bullet}) \cong \mathrm{Hom}^{\bullet}_{\mathcal{A}'}(X^{\bullet}, FY^{\bullet}).$$

**Theorem 14.14.** *The bi-∂-functor* $\mathrm{Hom}^{\bullet}_{\mathcal{A}} : \mathsf{K}^{*_1}(\mathcal{A})^{\mathrm{op}} \times \mathsf{K}^{*_2}(\mathcal{A}) \to \mathsf{K}(\mathfrak{Ab})$ *has the right derived functor* $\boldsymbol{R}^{*_1,*_2}\mathrm{Hom}^{\bullet}_{\mathcal{A}} : \mathrm{Hom}^{\bullet}_{\mathcal{A}} : \mathsf{D}^{*_1}(\mathcal{A})^{\mathrm{op}} \times \mathsf{D}^{*_2}(\mathcal{A}) \to \mathsf{D}(\mathfrak{Ab})$ *if it satisfies the following.*

1. *If* $\mathcal{A}$ *has enough projectives, then*
   $\boldsymbol{R}^{-,\infty}\mathrm{Hom}^{\bullet}_{\mathcal{A}}$ *exits and* $\boldsymbol{R}^{-,\infty}\mathrm{Hom}^{\bullet}_{\mathcal{A}} \cong \boldsymbol{R}^{-,\infty}_{\mathrm{II}}\boldsymbol{R}^{-,\infty}_{\mathrm{I}}\mathrm{Hom}^{\bullet}_{\mathcal{A}}$
2. *If* $\mathcal{A}$ *has enough injectives, then*
   $\boldsymbol{R}^{\infty,+}\mathrm{Hom}^{\bullet}_{\mathcal{A}}$ *exits and* $\boldsymbol{R}^{\infty,+}\mathrm{Hom}^{\bullet}_{\mathcal{A}} \cong \boldsymbol{R}^{\infty,+}_{\mathrm{I}}\boldsymbol{R}^{\infty,+}_{\mathrm{II}}\mathrm{Hom}^{\bullet}_{\mathcal{A}}$
3. *If* $\mathcal{A}$ *satisfies the condition Ab4 with enough projectives, then*
   $\boldsymbol{R}\mathrm{Hom}^{\bullet}_{\mathcal{A}}$ *exits and* $\boldsymbol{R}\mathrm{Hom}^{\bullet}_{\mathcal{A}} \cong \boldsymbol{R}_{\mathrm{II}}\boldsymbol{R}_{\mathrm{I}}\mathrm{Hom}^{\bullet}_{\mathcal{A}}$
4. *If* $\mathcal{A}$ *satisfies the condition Ab4* with enough injectives, then*
   $\boldsymbol{R}\mathrm{Hom}^{\bullet}_{\mathcal{A}}$ *exits and* $\boldsymbol{R}\mathrm{Hom}^{\bullet}_{\mathcal{A}} \cong \boldsymbol{R}_{\mathrm{I}}\boldsymbol{R}_{\mathrm{II}}\mathrm{Hom}^{\bullet}_{\mathcal{A}}$
5. *If* $\mathcal{A}$ *satisfies the conditions Ab4 and Ab4* with enough projectives and with enough injectives, then*
   $\boldsymbol{R}\mathrm{Hom}^{\bullet}_{\mathcal{A}} \cong \boldsymbol{R}_{\mathrm{II}}\boldsymbol{R}_{\mathrm{I}}\mathrm{Hom}^{\bullet}_{\mathcal{A}} \cong \boldsymbol{R}_{\mathrm{I}}\boldsymbol{R}_{\mathrm{II}}\mathrm{Hom}^{\bullet}_{\mathcal{A}}$

*Here* $\infty$ *means "nothing".*

*Proof.* By Lemma 14.9, Corollary 10.9, Propositions 14.6, 14.7 can be applied. □

**Remark 14.15.** In the above 5, for complexes $X^{\bullet}, Y^{\bullet} \in \mathsf{K}(\mathcal{A})$, we take $P^{\bullet} \in \mathsf{K}^{\mathrm{s}}(\mathsf{Proj}\,\mathcal{A})$ $I^{\bullet} \in \mathsf{K}^{\mathrm{s}}(\mathsf{Inj}\,\mathcal{A})$ which have quasi-isomorphisms $P^{\bullet} \to X^{\bullet}$, $Y^{\bullet} \to I^{\bullet}$. Then we have an isomophisms

$$\boldsymbol{R}\mathrm{Hom}^{\bullet}_{\mathcal{A}}(X^{\bullet}, Y^{\bullet}) \cong \mathrm{Hom}^{\bullet}_{\mathcal{A}}(P^{\bullet}, Y^{\bullet}) \cong \mathrm{Hom}^{\bullet}_{\mathcal{A}}(X^{\bullet}, I^{\bullet}) \cong \mathrm{Hom}^{\bullet}_{\mathcal{A}}(P^{\bullet}, I^{\bullet}).$$

**Corollary 14.16.** *Assume that* $\mathcal{A}$ *satisfies one of the conditions of Theorem 14.14. For* $X^{\bullet} \in \mathsf{D}^{*_1}(\mathcal{A})$, $Y^{\bullet} \in \mathsf{D}^{*_2}(\mathcal{A})$ *and* $n \in \mathbb{Z}$, *we have an isomorphism*

$$\mathrm{H}^n\,\boldsymbol{R}^{*_1,*_2}\mathrm{Hom}^{\bullet}_{\mathcal{A}}(X^{\bullet}, Y^{\bullet}) \cong \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(X^{\bullet}, Y^{\bullet}[n]).$$

*Proof.* By Proposition 11.10 and Lemma 10.10, for $X^{\bullet} \in \mathsf{D}^{*_1}(\mathcal{A})$, $Y^{\bullet} \in \mathsf{D}^{*_2}(\mathcal{A})$, we have either a quasi-isomorphism $P^{\bullet} \to X^{\bullet}$ or a quasi-isomorphism $Y^{\bullet} \to I^{\bullet}$ with $P^{\bullet} \in \mathsf{K}^{*_1}(\mathsf{Proj}\,\mathcal{A})$, $I^{\bullet} \in \mathsf{K}^{*_2}(\mathsf{Proj}\,\mathcal{A})$. According to Corollary 10.9 and Proposition 11.12, we have one of isomorphisms

$$\begin{aligned}\mathrm{H}^n\,\boldsymbol{R}^{*_1,*_2}\mathrm{Hom}^{\bullet}_{\mathcal{A}}(X^{\bullet}, Y^{\bullet}) &\cong \mathrm{H}^n\,\mathrm{Hom}^{\bullet}_{\mathcal{A}}(P^{\bullet}, Y^{\bullet})\\ &\cong \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(P^{\bullet}, Y^{\bullet}[n])\\ &\cong \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(X^{\bullet}, Y^{\bullet}[n]),\end{aligned}$$

$$\begin{aligned}\mathrm{H}^n\,\boldsymbol{R}^{*_1,*_2}\mathrm{Hom}^{\bullet}_{\mathcal{A}}(X^{\bullet}, Y^{\bullet}) &\cong \mathrm{H}^n\,\mathrm{Hom}^{\bullet}_{\mathcal{A}}(X^{\bullet}, Y^{\bullet})\\ &\cong \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X^{\bullet}, I^{\bullet}[n])\\ &\cong \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(X^{\bullet}, Y^{\bullet}[n]).\end{aligned}$$

□

**Definition 14.17.** For a complex $X^\cdot$ of right $A$-modules and a complex $Y^\cdot$ of left $A$-modules we define the double complex $X^\cdot \ddot{\otimes}_A Y^\cdot$ by

$$\begin{aligned} X^\cdot \overset{p,q}{\otimes}_A Y^\cdot &= X^p \otimes_A Y^q \\ d^{p,q}_{\mathrm{I}\, X^\cdot \ddot{\otimes}_A Y^\cdot} &= d^p_X \otimes_A Y^q \\ d^{p,q}_{\mathrm{II}\, X^\cdot \ddot{\otimes}_A Y^\cdot} &= (-1)^{p+q} X^p \otimes_A d^q_Y \end{aligned}$$

and define the complex $X^\cdot \dot{\otimes}_A Y^\cdot$ by

$$\mathrm{Tot} X^\cdot \ddot{\otimes}_A Y^\cdot.$$

Then it is easy to see that

$$\dot{\otimes}_A : \mathsf{C}(\mathsf{Mod}\, A) \times \mathsf{C}(\mathsf{Mod}\, A^{\mathrm{op}}) \to \mathsf{C}(\mathfrak{Ab})$$

is a bifunctor.

**Lemma 14.18.** *For a complexes $X^\cdot \in \mathsf{C}(\mathsf{Mod}\, A), Y^\cdot \in \mathsf{C}(\mathsf{Mod}\, A^{\mathrm{op}})$, the following hold.*

1. $d_{(TX^\cdot)\dot{\otimes}_A Y^\cdot} = -Td_{X^\cdot \dot{\otimes}_A Y^\cdot}$
2. $d_{X^\cdot \dot{\otimes}_A TY^\cdot} = Td_{X^\cdot \dot{\otimes}_A Y^\cdot}$
3. *Define $\theta_1^{p,q} : (TX^\cdot) \overset{p,q}{\otimes}_A Y^\cdot \to X^\cdot \overset{p+1,q}{\otimes}_A Y^\cdot$ by the identities, then we have an isomorphism*
$$\theta_1 : \dot{\otimes}_A \circ (T \times \mathbf{1}_{\mathsf{C}(\mathsf{Mod}\, A)}) \xrightarrow{\sim} T \circ \dot{\otimes}_A.$$
4. *Define $\theta_2^{p,q} : X^\cdot \overset{p,q}{\otimes}_A TY^\cdot \to X^\cdot \overset{p,q+1}{\otimes}_A Y^\cdot$ by the $(-1)^{p+q}$, then we have an isomorphism*
$$\theta_2 : \dot{\otimes}_A \circ (\mathbf{1}_{\mathsf{C}(\mathsf{Mod}\, A^{\mathrm{op}})} \times T) \xrightarrow{\sim} T \circ \dot{\otimes}_A.$$

**Lemma 14.19.** *The following hold.*

1. *For a morphism $u : X^\cdot \to Y^\cdot$ in $\mathsf{C}(\mathsf{Mod}\, A^{\mathrm{op}})$ and $N^\cdot \in \mathsf{C}(\mathsf{Mod}\, A)$ we have $N^\cdot \dot{\otimes}_A \mathrm{M}^\cdot(u) \cong \mathrm{M}^\cdot(N^\cdot \dot{\otimes}_A u)$.*
2. *For a morphism $u : X^\cdot \to Y^\cdot$ in $\mathsf{C}(\mathsf{Mod}\, A)$ and $N^\cdot \in \mathsf{C}(\mathsf{Mod}\, A^{\mathrm{op}})$ we have $\mathrm{M}^\cdot(u) \dot{\otimes}_A N^\cdot \cong \mathrm{M}^\cdot(u \dot{\otimes}_A N^\cdot)$.*

**Proposition 14.20.** *The bi-functor $\dot{\otimes}_A : \mathsf{C}(\mathsf{Mod}\, A) \times \mathsf{C}(\mathsf{Mod}\, A^{\mathrm{op}}) \to \mathsf{C}(\mathfrak{Ab})$ induces the bi-$\partial$-functor*

$$\dot{\otimes}_A : \mathsf{K}(\mathsf{Mod}\, A) \times \mathsf{K}(\mathsf{Mod}\, A^{\mathrm{op}}) \to \mathsf{K}(\mathfrak{Ab}).$$

**Lemma 14.21.** *Let $D_{\mathbb{Z}} = \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z}) : \mathsf{Mod}\, A \to \mathsf{Mod}\, A^{\mathrm{op}}$ (resp., $D_{\mathbb{Z}} = \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z}) : \mathsf{Mod}\, A^{\mathrm{op}} \to \mathsf{Mod}\, A$). Then the following hold.*

1. *For an sequence $X \to Y \to Z$ of $A$-modules, $X \to Y \to Z$ is exact if and only if $D_{\mathbb{Z}}Z \to D_{\mathbb{Z}}Y \to D_{\mathbb{Z}}X$ is exact.*
2. *For an $A$-module $M$, $M$ is a flat $A$-module if and only if $D_{\mathbb{Z}}M$ is an injective $A$-module.*

**Theorem 14.22.** *The bi-∂-functor* $\dot{\otimes}_A : \mathsf{K}(\mathsf{Mod}\, A) \times \mathsf{K}(\mathsf{Mod}\, A^{\mathrm{op}}) \to \mathsf{K}(\mathfrak{Ab})$ *has the left derived functor*

$$\dot{\otimes}^{\boldsymbol{L}}_A : \mathsf{D}(\mathsf{Mod}\, A) \times \mathsf{D}(\mathsf{Mod}\, A^{\mathrm{op}}) \to \mathsf{D}(\mathfrak{Ab}).$$

*Proof.* For $X^{\boldsymbol{\cdot}} \in \mathsf{K}(\mathsf{Mod}\, A)$, $Y^{\boldsymbol{\cdot}} \in \mathsf{K}(\mathsf{Mod}\, A^{\mathrm{op}})$, we have isomorphisms (see Proposition 15.4)

$$\begin{aligned} D_{\mathbb{Z}}(X^{\boldsymbol{\cdot}} \dot{\otimes}_A Y^{\boldsymbol{\cdot}}) &= D_{\mathbb{Z}}(\mathrm{Tot}\, X^{\boldsymbol{\cdot}} \ddot{\otimes}_A Y^{\boldsymbol{\cdot}}) \\ &\cong \overset{\wedge}{\mathrm{Tot}} D_{\mathbb{Z}}(X^{\boldsymbol{\cdot}} \ddot{\otimes}_A Y^{\boldsymbol{\cdot}}) \\ &\cong \overset{\wedge}{\mathrm{Tot}}\, \mathrm{Hom}^{\boldsymbol{\cdot\cdot}}_{A^{\mathrm{op}}}(X^{\boldsymbol{\cdot}}, D_{\mathbb{Z}} Y^{\boldsymbol{\cdot}}) \\ &\cong \mathrm{Hom}^{\boldsymbol{\cdot}}_{A^{\mathrm{op}}}(X^{\boldsymbol{\cdot}}, D_{\mathbb{Z}} Y^{\boldsymbol{\cdot}}). \end{aligned}$$

By Lemma 14.9, we can apply Proposition 11.12 to the above isomorphism. Then by Lemma 14.21, for $P^{\boldsymbol{\cdot}} \in \mathsf{K}^{\mathrm{s}}(\mathsf{Proj}\, A)$ and $Y^{\boldsymbol{\cdot}} \in \mathsf{K}(\mathsf{Mod}\, A^{\mathrm{op}})$, $P^{\boldsymbol{\cdot}} \dot{\otimes}_A Y^{\boldsymbol{\cdot}}$ are acyclic either if $P^{\boldsymbol{\cdot}}$ is acyclic or if $Y^{\boldsymbol{\cdot}}$ is acyclic. According to the left derived version of Theorem 12.5, we complete the proof. □

**Remark 14.23.** In the above, for complexes $X^{\boldsymbol{\cdot}} \in \mathsf{K}(\mathsf{Mod}\, A)$, $Y^{\boldsymbol{\cdot}} \in \mathsf{K}(\mathsf{Mod}\, A^{\mathrm{op}})$, we take $P^{\boldsymbol{\cdot}} \in \mathsf{K}^{\mathrm{s}}(\mathsf{Proj}\, A)$ $Q^{\boldsymbol{\cdot}} \in \mathsf{K}^{\mathrm{s}}(\mathsf{Proj}\, A^{\mathrm{op}})$ which have quasi-isomorphisms $P^{\boldsymbol{\cdot}} \to X^{\boldsymbol{\cdot}}$, $Q^{\boldsymbol{\cdot}} \to Y^{\boldsymbol{\cdot}}$. Then we have an isomophisms

$$X^{\boldsymbol{\cdot}} \dot{\otimes}^{\boldsymbol{L}}_A Y^{\boldsymbol{\cdot}} \cong P^{\boldsymbol{\cdot}} \dot{\otimes}_A Y^{\boldsymbol{\cdot}} \cong X^{\boldsymbol{\cdot}} \dot{\otimes}_A Q^{\boldsymbol{\cdot}} \cong P^{\boldsymbol{\cdot}} \dot{\otimes}_A Q^{\boldsymbol{\cdot}}.$$

For $X \in \mathsf{Mod}\, A$ and $Y \in \mathsf{Mod}\, A^{\mathrm{op}}$, we denote $\mathrm{Tor}^i_A(X, Y) = \mathrm{H}^i(X \dot{\otimes}^{\boldsymbol{L}}_A Y)$.

## 15. Bimodule Complexes

Throughout this section, $k$ is a commutative ring, $A, B, C$ are $k$-algebras, ${}_AU_B$ an $A$-$B$-bimodule, ${}_BV_C$ an $B$-$C$-bimodule, ${}_AW_C$ an $A$-$C$-bimodule and ${}_CS_A$ a $C$-$A$-bimodule.

**Proposition 15.1.** *The following hold.*

1. $\mathrm{Hom}_{A^{\mathrm{op}} \otimes_k B}({}_AU_B, \mathrm{Hom}_C({}_BV_C, {}_AW_C)) \cong \mathrm{Hom}_{A^{\mathrm{op}} \otimes_k C}({}_AU \otimes_B V_C, {}_AW_C)$.
2. $\mathrm{Hom}_{A^{\mathrm{op}} \otimes_k B}({}_AU_B, \mathrm{Hom}_C({}_BV_C, {}_AW_C)) \cong \mathrm{Hom}_{B \otimes_k C^{\mathrm{op}}}({}_BV_C, \mathrm{Hom}_A({}_AU_B, {}_AW_C))$.
3. $({}_AU \otimes_B V_C) \otimes_{A^{\mathrm{op}} \otimes_k C} ({}_CS_A) \cong ({}_AU_B) \otimes_{A^{\mathrm{op}} \otimes_k B} ({}_BV \otimes_C S_A)$.
4. *If* ${}_AU_B$ *is* $A^{\mathrm{op}} \otimes_k B$*-projective,* $V_C$ *is* $C$*-projective, then* ${}_AU \otimes_B V_C$ *is* $A^{\mathrm{op}} \otimes_k C$*-projective.*
5. *If* ${}_BV$ *is* $B$*-flat,* ${}_AW_C$ *is* $A^{\mathrm{op}} \otimes_k C$*-injective, then* $\mathrm{Hom}_C({}_BV_C, {}_AW_C)$ *is* $A^{\mathrm{op}} \otimes_k B$*-injective.*
6. *If* ${}_BV_C$ *is* $B^{\mathrm{op}} \otimes_k C$*-projective,* ${}_AW$ *is* $A$*-injective, then* $\mathrm{Hom}_C({}_BV_C, {}_AW_C)$ *is* $A^{\mathrm{op}} \otimes_k B$*-injective.*
7. *If* ${}_AU_B$ *is* $A^{\mathrm{op}} \otimes_k B$*-flat,* $V_C$ *is* $C$*-flat , then* ${}_AU \otimes_B V_C$ *is* $A^{\mathrm{op}} \otimes_k C$*-flat.*

**Corollary 15.2.** *The following hold.*

1. ${}_AU$ *is* $A$*-projective,* $V_C$ *is* $C$*-projective, then* ${}_AU \otimes_k V_C$ *is* $A^{\mathrm{op}} \otimes_k C$*-projective.*
2. ${}_BV$ *is* $B$*-flat,* ${}_AW$ *is* $A$*-injective, then* $\mathrm{Hom}_k({}_BV_k, {}_AW_k)$ *is* $A^{\mathrm{op}} \otimes_k B$*-injective.*
3. ${}_AU$ *is* $A$*-flat,* $V_C$ *is* $C$*-flat, then* ${}_AU \otimes_k V_C$ *is* $A^{\mathrm{op}} \otimes_k C$*-flat.*

**Proposition 15.3.** *Let* ${}_AM$ *be an* $A$*-module,* ${}_BN$ *a* $B$*-module, then the following hold.*

1. $\mathrm{Hom}_{A^{\mathrm{op}} \otimes_k B}({}_AU_B, \mathrm{Hom}_k({}_BB_k, {}_AV_k)) \cong \mathrm{Hom}_A({}_AU, {}_AV)$.

2. $\mathrm{Hom}_A({}_AM, {}_AW) \cong \mathrm{Hom}_{A^{\mathrm{op}}\otimes_k C}({}_AM\otimes_k C_C, {}_AW_C)$.
3. $U\otimes_B N \cong ({}_AU_B)\otimes_{A^{\mathrm{op}}\otimes_k B}({}_BN\otimes_k A_A)$.
4. *If $B$ is $k$-projective, ${}_AU_B$ is $A^{\mathrm{op}}\otimes_k B$-projective, then ${}_AU$ is $A$-projective.*
5. *$C$ is $k$-flat, ${}_AW_C$ is $A^{\mathrm{op}}\otimes_k C$-injective, then ${}_AW$ is $A$-injective.*
6. *$B$ is $k$-flat, ${}_AU_B$ is $A^{\mathrm{op}}\otimes_k B$-flat, then ${}_AU$ is $A$-flat.*

**Proposition 15.4.** *For $U^{\cdot} \in \mathsf{K}(\mathsf{Mod}\, A^{\mathrm{op}}\otimes_k B)$, $V^{\cdot} \in \mathsf{K}(\mathsf{Mod}\, B\otimes_k C^{\mathrm{op}})$, $W^{\cdot} \in \mathsf{K}(\mathsf{Mod}\, A^{\mathrm{op}}\otimes_k C)$, $S^{\cdot} \in \mathsf{K}(\mathsf{Mod}\, C\otimes_k A^{\mathrm{op}})$, the following hold.*

1. *We have an isomorphism*

$$\mathrm{Hom}^{\cdot}_{A^{\mathrm{op}}\otimes_k B}({}_AU^{\cdot}_B, \mathrm{Hom}^{\cdot}_C({}_BV^{\cdot}_C, {}_AW^{\cdot}_C)) \cong \mathrm{Hom}^{\cdot}_{A^{\mathrm{op}}\otimes_k C}({}_AU^{\cdot} \dot{\otimes}_B V^{\cdot}_C, {}_AW^{\cdot}_C).$$

2. *We have an isomorphism*

$$\mathrm{Hom}^{\cdot}_{A^{\mathrm{op}}\otimes_k B}({}_AU^{\cdot}_B, \mathrm{Hom}^{\cdot}_C({}_BV^{\cdot}_C, {}_AW^{\cdot}_C)) \cong \mathrm{Hom}^{\cdot}_{B\otimes_k C^{\mathrm{op}}}({}_BV^{\cdot}_C, \mathrm{Hom}^{\cdot}_A({}_AU^{\cdot}_B, {}_AW^{\cdot}_C)).$$

3. *We have an isomorphism*

$$({}_AU^{\cdot} \dot{\otimes}_B V^{\cdot}_C) \dot{\otimes}_{A^{\mathrm{op}}\otimes_k C} ({}_CS^{\cdot}_A) \cong ({}_AU^{\cdot}_B) \dot{\otimes}_{A^{\mathrm{op}}\otimes_k B} ({}_BV^{\cdot} \dot{\otimes}_C S^{\cdot}_A).$$

*Proof.* 1. Let $\alpha$ be the trifunctorial isomorphism in 1 of Proposition 15.1. For every $(p, q, r) \in \mathbb{Z}^3$, define

$$\begin{aligned}\phi_{p,q,r} = (-1)^{\frac{r(2q+r+1)}{2}}\alpha : \mathrm{Hom}_{A^{\mathrm{op}}\otimes_k B}({}_AU^{-p}_B, \mathrm{Hom}_C({}_BV^{-q}_C, {}_AW^{r}_C)) \to \\ \mathrm{Hom}_{A^{\mathrm{op}}\otimes_k C}({}_AU^{-p}\otimes_B V^{-q}_C, {}_AW^{r}_C),\end{aligned}$$

then $(\phi_{p,q,r})$ induces the isomorphism between triple complexes. By taking $\overset{\wedge}{\mathrm{Tot}}$, we have the assertion.

2. Let $\beta$ be the trifunctorial isomorphism in 2 of Proposition 15.1. For every $(p, q, r) \in \mathbb{Z}^3$, define

$$\begin{aligned}{\phi'}_{p,q,r} = (-1)^{\frac{(p+q)(p+q+2r+1)}{2}}\beta : \mathrm{Hom}_{A^{\mathrm{op}}\otimes_k B}({}_AU^{-p}_B, \mathrm{Hom}_C({}_BV^{-q}_C, {}_AW^{r}_C)) \to \\ \mathrm{Hom}_{B^{\mathrm{op}}\otimes_k C}({}_BV^{-q}_C, \mathrm{Hom}_A({}_AU^{-p}_B, {}_AW^{r}_C)),\end{aligned}$$

then $(\phi_{p,q,r})$ induces the isomorphism between triple complexes. By taking $\overset{\wedge}{\mathrm{Tot}}$, we have the assertion.

3. Let $\gamma$ be the trifunctorial isomorphism in 3 of Proposition 15.1. For every $(p, q, r) \in \mathbb{Z}^3$, define

$$\begin{aligned}\psi_{p,q,r} = (-1)^{\frac{r(2q+r-1)}{2}}\gamma : ({}_AU^{p}\otimes_B V^{q}_C)\otimes_{A^{\mathrm{op}}\otimes_k C}({}_CS^{r}_A) \to \\ ({}_AU^{p}_B)\otimes_{A^{\mathrm{op}}\otimes_k B}({}_BV^{q}\otimes_C S^{r}_A),\end{aligned}$$

then $(\phi_{p,q,r})$ induces the isomorphism between triple complexes. By taking Tot, we have the assertion.

□

**Proposition 15.5.** *The following hold.*

1. *For a functor $- \dot{\otimes}_A U^{\cdot}_B : \mathsf{K}(\mathsf{Mod}\, A) \to \mathsf{K}(\mathsf{Mod}\, B)$ and its right adjoint $\mathrm{Hom}^{\cdot}_B({}_AU^{\cdot}_B, -) : \mathsf{K}(\mathsf{Mod}\, B) \to \mathsf{K}(\mathsf{Mod}\, A)$, there exist the left derived functor $- \dot{\otimes}^{\boldsymbol{L}}_A U^{\cdot}_B : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, B)$ and the right derived functor $\boldsymbol{R}\mathrm{Hom}^{\cdot}_B({}_AU^{\cdot}_B, -) : \mathsf{D}(\mathsf{Mod}\, B) \to \mathsf{D}(\mathsf{Mod}\, A)$ such that*

$$\boldsymbol{R}\mathrm{Hom}^{\cdot}_B(- \dot{\otimes}^{\boldsymbol{L}}_A U^{\cdot}_B, ?) \cong \boldsymbol{R}\mathrm{Hom}^{\cdot}_A(-, \boldsymbol{R}\mathrm{Hom}^{\cdot}_B({}_AU^{\cdot}_B, ?)).$$

*In particular, we have* $-\dot{\otimes}^{\boldsymbol{L}}_A U^{\cdot}_B \dashv \boldsymbol{R}\mathrm{Hom}^{\cdot}_B({}_AU^{\cdot}_B, ?)$.

2. *For a right adjoint pair of functors* $\mathrm{Hom}^{\cdot}_A(-, {}_AU^{\cdot}_B) : \mathsf{K}(\mathsf{Mod}\, A^{\mathrm{op}}) \to \mathsf{K}(\mathsf{Mod}\, B)$, $\mathrm{Hom}^{\cdot}_B(-, {}_AU^{\cdot}_B) : \mathsf{K}(\mathsf{Mod}\, B) \to \mathsf{K}(\mathsf{Mod}\, A^{\mathrm{op}})$, *there exist the right derived functors* $\boldsymbol{R}\mathrm{Hom}^{\cdot}_A(-, {}_AU^{\cdot}_B) : \mathsf{D}(\mathsf{Mod}\, A^{\mathrm{op}}) \to \mathsf{D}(\mathsf{Mod}\, B)$, $\boldsymbol{R}\mathrm{Hom}^{\cdot}_B(-, {}_AU^{\cdot}_B) : \mathsf{D}(\mathsf{Mod}\, B) \to \mathsf{D}(\mathsf{Mod}\, A^{\mathrm{op}})$ *such that*

$$\boldsymbol{R}\mathrm{Hom}^{\cdot}_B(-, \boldsymbol{R}\mathrm{Hom}^{\cdot}_A(?, {}_AU^{\cdot}_B)), \cong \boldsymbol{R}\mathrm{Hom}^{\cdot}_A(?, \boldsymbol{R}\mathrm{Hom}^{\cdot}_B(-, {}_AU^{\cdot}_B)).$$

*In particular,* ($\boldsymbol{R}\mathrm{Hom}^{\cdot}_A(?, {}_AU^{\cdot}_B)$, $\boldsymbol{R}\mathrm{Hom}^{\cdot}_B(-, {}_AU^{\cdot}_B)$) *is a right adjoint pair.*

*Proof.* 1. Let $X^{\cdot} \in \mathsf{K}(\mathsf{Mod}\, A^{\mathrm{op}})$, $Y^{\cdot} \in \mathsf{K}(\mathsf{Mod}\, B)$. According to Theorems 14.14, 14.22 and Remark 14.15, we may assume $X^{\cdot} \in \mathsf{K}^{\mathrm{s}}(\mathsf{Proj}\, A)$, $Y^{\cdot} \in \mathsf{K}^{\mathrm{s}}(\mathsf{Inj}\, B)$. It is clear for the existence of the derived functor. By Proposition 15.4 we have isomorphisms

$$\begin{aligned}\boldsymbol{R}\mathrm{Hom}^{\cdot}_B(X^{\cdot} \dot{\otimes}^{\boldsymbol{L}}_A U^{\cdot}_B, Y^{\cdot}) &\cong \mathrm{Hom}^{\cdot}_B(X^{\cdot} \dot{\otimes}_A U^{\cdot}_B, Y^{\cdot}) \\ &\cong \mathrm{Hom}^{\cdot}_A(X^{\cdot}, \mathrm{Hom}^{\cdot}_B({}_AU^{\cdot}_B, Y^{\cdot})) \\ &\cong \boldsymbol{R}\mathrm{Hom}^{\cdot}_A(X^{\cdot}, \boldsymbol{R}\mathrm{Hom}^{\cdot}_B({}_AU^{\cdot}_B, Y^{\cdot})).\end{aligned}$$

We get the last assertion by taking cohomologies of the above isomorphisms.

2. Let $X^{\cdot} \in \mathsf{K}(\mathsf{Mod}\, A^{\mathrm{op}})$, $Y^{\cdot} \in \mathsf{K}(\mathsf{Mod}\, B)$. According to Theorem 14.14 and Remark 14.15, we may assume $X^{\cdot} \in \mathsf{K}^{\mathrm{s}}(\mathsf{Proj}\, A^{\mathrm{op}})$, $Y^{\cdot} \in \mathsf{K}^{\mathrm{s}}(\mathsf{Proj}\, B)$. It is clear for the existence of the derived functor. By Proposition 15.4 we have isomorphisms

$$\begin{aligned}\boldsymbol{R}\mathrm{Hom}^{\cdot}_B(Y^{\cdot}, \boldsymbol{R}\mathrm{Hom}^{\cdot}_A(X^{\cdot}, {}_AU^{\cdot}_B)) &\cong \mathrm{Hom}^{\cdot}_B(Y^{\cdot}, \mathrm{Hom}^{\cdot}_A(X^{\cdot}, {}_AU^{\cdot}_B)) \\ &\cong \mathrm{Hom}^{\cdot}_A(X^{\cdot}, \boldsymbol{R}\mathrm{Hom}^{\cdot}_B(Y^{\cdot}, {}_AU^{\cdot}_B)) \\ &\cong \boldsymbol{R}\mathrm{Hom}^{\cdot}_A(X^{\cdot}, \boldsymbol{R}\mathrm{Hom}^{\cdot}_B(Y^{\cdot}, {}_AU^{\cdot}_B)).\end{aligned}$$

We get the last assertion by taking cohomologies of the above isomorphisms. □

**Remark 15.6.** The above derived functors $-\dot{\otimes}^{\boldsymbol{L}}_A U^{\cdot}_B : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, B)$ and $\boldsymbol{R}\mathrm{Hom}^{\cdot}_B({}_AU^{\cdot}_B, -) : \mathsf{D}(\mathsf{Mod}\, B) \to \mathsf{D}(\mathsf{Mod}\, A)$ are the derived functors of $\partial$-functors $-\dot{\otimes}_A U^{\cdot}_B$ and $\mathrm{Hom}^{\cdot}_B({}_AU^{\cdot}_B, -)$, respectively. But they are not the derived functors of bi-$\partial$-functors in general!

We denote by $Res_A : \mathsf{Mod}\, A^{\mathrm{op}} \otimes_k B \to \mathsf{Mod}\, A$ the forgetful functor, and use the same symbol $Res_A : \mathsf{K}(\mathsf{Mod}\, A^{\mathrm{op}} \otimes_k B) \to \mathsf{K}(\mathsf{Mod}\, A)$ for the induced $\partial$-functor.

**Proposition 15.7.** *If $B$ is $k$-projective or $C$ is $k$-flat, then*

$$\boldsymbol{R}\mathrm{Hom}^{\cdot}_A : \mathsf{D}(\mathsf{Mod}\, B^{\mathrm{op}} \otimes_k A)^{\mathrm{op}} \times \mathsf{D}(\mathsf{Mod}\, C^{\mathrm{op}} \otimes_k A) \to \mathsf{D}(\mathsf{Mod}\, C^{\mathrm{op}} \otimes_k B)$$

*exists, and we have a commutative diagram*

$$\begin{array}{ccc}\mathsf{D}(\mathsf{Mod}\, B^{\mathrm{op}} \otimes_k A)^{\mathrm{op}} \times \mathsf{D}(\mathsf{Mod}\, C^{\mathrm{op}} \otimes_k A) & \xrightarrow{Res_A^{\mathrm{op}} \times Res_A} & \mathsf{D}(\mathsf{Mod}\, A)^{\mathrm{op}} \times \mathsf{D}(\mathsf{Mod}\, A) \\ {\scriptstyle \boldsymbol{R}\mathrm{Hom}^{\cdot}_A}\downarrow & & \downarrow{\scriptstyle \boldsymbol{R}\mathrm{Hom}^{\cdot}_A} \\ \mathsf{D}(\mathsf{Mod}\, C^{\mathrm{op}} \otimes_k B) & \xrightarrow{Res_k} & \mathsf{D}(\mathsf{Mod}\, k)\end{array}$$

*Proof.* Assume $B$ is $k$-projective. According to Proposition 15.4, for ${}_BX^{\cdot}_A \in \mathsf{K}(\mathsf{Mod}\, B \otimes_k A)$, $Y^{\cdot}_A \in \mathsf{K}(\mathsf{Mod}\, A)$, we have

$$\mathrm{Hom}^{\cdot}_{B^{\mathrm{op}} \otimes_k A}({}_BX^{\cdot}_A, \mathrm{Hom}^{\cdot}_k({}_kB_B, {}_kY^{\cdot}_A)) \cong \mathrm{Hom}^{\cdot}_A(X^{\cdot}_A, Y^{\cdot}_A).$$

If ${}_BX^{\cdot}_A \in \mathsf{K}^{\mathrm{s}}(\mathsf{Proj}\, B^{\mathrm{op}} \otimes_k A)$, then by the above isomorphism, we have $Res_A X^{\cdot} \in \mathsf{K}^{\mathrm{s}}(\mathsf{Proj}\, A)$. Therefore we get the assertion by Theorem 14.5.

Assume $C$ is $k$-flat. For ${}_CY^{\cdot}_A \in \mathsf{K}(\mathsf{Mod}\, C^{\mathrm{op}} \otimes_k A)$, $X^{\cdot}_A \in \mathsf{K}(\mathsf{Mod}\, A)$, we have

$$\mathrm{Hom}^{\cdot}_{C^{\mathrm{op}}\otimes_k A}({}_CC \otimes_k X^{\cdot}_A, {}_CY^{\cdot}_A) \cong \mathrm{Hom}^{\cdot}_A(X^{\cdot}_A, Y^{\cdot}_A).$$

If ${}_CY^{\cdot}_A \in \mathsf{K}^{\mathrm{s}}(\mathsf{Inj}\, C^{\mathrm{op}} \otimes_k A)$, then by the above isomorphism, we have $Res_A Y^{\cdot} \in \mathsf{K}^{\mathrm{s}}(\mathsf{Inj}\, A)$. By the same reason as the above. □

**Proposition 15.8.** *If either $A$ or $C$ is $k$-flat, then*

$$\dot{\otimes}^{\boldsymbol{L}}_A : \mathsf{D}(\mathsf{Mod}\, A^{\mathrm{op}}\otimes_k B) \times \mathsf{D}(\mathsf{Mod}\, B^{\mathrm{op}} \otimes_k C) \to \mathsf{D}(\mathsf{Mod}\, A^{\mathrm{op}} \otimes_k C)$$

*exists, and we have a commutative diagram*

$$\begin{array}{ccc} \mathsf{D}(\mathsf{Mod}\, A^{\mathrm{op}}\otimes_k B) \times \mathsf{D}(\mathsf{Mod}\, B^{\mathrm{op}} \otimes_k C) & \xrightarrow{Res_B \times Res_{B^{\mathrm{op}}}} & \mathsf{D}(\mathsf{Mod}\, B) \times \mathsf{D}(\mathsf{Mod}\, B^{\mathrm{op}}) \\ \dot{\otimes}^{\boldsymbol{L}}_B \downarrow & & \downarrow \dot{\otimes}^{\boldsymbol{L}}_B \\ \mathsf{D}(\mathsf{Mod}\, A^{\mathrm{op}} \otimes_k C) & \xrightarrow{Res_k} & \mathsf{D}(\mathsf{Mod}\, k) \end{array}$$

*Proof.* Assume $A$ is $k$-flat. For $X^{\cdot} \in \mathsf{K}(\mathsf{Mod}\, A^{\mathrm{op}}\otimes_k B)$, $Y^{\cdot} \in \mathsf{K}(\mathsf{Mod}\, B^{\mathrm{op}})$, by Proposition 15.4, we have an isomorphism

$$({}_AX^{\cdot}_B) \dot{\otimes}_{A^{\mathrm{op}}\otimes B} ({}_BY^{\cdot}\otimes_k A_A) \cong X^{\cdot} \dot{\otimes}_B Y^{\cdot}.$$

If ${}_AX^{\cdot}_B \in \mathsf{K}^{\mathrm{s}}(\mathsf{Proj}\, A^{\mathrm{op}} \otimes_k B)$, then by the proof of Theorem 14.22, $X^{\cdot} \dot{\otimes}_B Y^{\cdot}$ is acyclic if either $X^{\cdot}$ or $Y^{\cdot}$ is acyclic. Therefore we get the assertion by Theorem 14.5. In case of $C$ being $k$-flat, similarly. □

**Example 15.9.** Let $F$ be a field, $k = A = F[[x]]$, $B = C = F[[x]]/(x^2)$. Let ${}_AM_B = F[[x]]/(x^2)$, ${}_AN_C = F$, $\boldsymbol{R}'\,\mathrm{Hom}^{\cdot}_A$ the right derived functor of

$$\mathrm{Hom}^{\cdot}_A : \mathsf{K}^-(\mathsf{Mod}\, B^{\mathrm{op}}\otimes_k A)^{\mathrm{op}} \times \mathsf{K}^+(\mathsf{Mod}\, C^{\mathrm{op}}\otimes_k A) \to \mathsf{K}(\mathsf{Mod}\, C^{\mathrm{op}}\otimes_k B).$$

Then we have

$$\boldsymbol{R}'\,\mathrm{Hom}^{\cdot}_A(M, N) \cong \mathrm{Hom}_A(F[[x]]/(x^2), F).$$

Let $\boldsymbol{R}\,\mathrm{Hom}^{\cdot}_A$ be the right derived functor of $\mathrm{Hom}^{\cdot}_A : \mathsf{K}^-(\mathsf{Mod}\, A)^{\mathrm{op}} \times \mathsf{K}^+(\mathsf{Mod}\, A) \to \mathsf{K}(\mathsf{Mod}\, k)$. Then we have

$$\boldsymbol{R}\,\mathrm{Hom}^{\cdot}_A(M, N) \cong X^0 \to X^1$$

where $X^0 \to X^1 = \mathrm{Hom}_A(F[[x]], F) \xrightarrow{\mathrm{Hom}_A(x^2, F)} \mathrm{Hom}_A(F[[x]], F)$. Then

$$\boldsymbol{R}'\,\mathrm{Hom}^{\cdot}_A(M, N) \ncong \boldsymbol{R}\,\mathrm{Hom}^{\cdot}_A(M, N)$$

in $\mathsf{D}(\mathsf{Mod}\, k)$.

## 16. Tilting Complexes

Throughout this section, $A, B, C$ are rings. We recall that $\mathsf{mod}\, A$ is the category of finitely presented right $A$-modules, and that $\mathsf{proj}\, A$ is the full subcategory of $\mathsf{mod}\, A$ consisting of finitely generated projective right $A$-modules.

**Definition 16.1** (Perfect Complex)**.** A complex $X^{\cdot} \in \mathsf{D}(\mathsf{Mod}\, A)$ is called *a perfect complex* if $X^{\cdot}$ is isomorphic to a complex of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ in $\mathsf{D}(\mathsf{Mod}\, A)$. We denote by $\mathsf{D}(\mathsf{Mod}\, A)_{\mathrm{perf}}$ the triangulated full subcategory of $\mathsf{D}(\mathsf{Mod}\, A)$ consisting of perfect complexes.

**Lemma 16.2.** *For $X^{\cdot} \in \mathsf{K}^{\mathrm{b}}(\mathsf{Proj}\, A)$, the following are equivalent.*

1. *$X^{\cdot}$ is a compact object in $\mathsf{K}^{\mathrm{b}}(\mathsf{Proj}\, A)$.*
2. *$X^{\cdot}$ is isomorphic to an object of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$.*

*Proof.* $2 \Rightarrow 1$. By Lemma 16.3.

$1 \Rightarrow 2$. Let $X^{\cdot} = X^0 \xrightarrow{d^0} X^1 \to \ldots \to X^n$, with $X^i \in \mathsf{Proj}\, A$. By adding $P \xrightarrow{1} P$ to $X^{\cdot}$, we may assume that $X^0$ is a free $A$-module $A^{(I)}$. If $I$ is a finite set, then by $2 \Rightarrow 1$ $X^0$ is also compact, and hence $\tau_{\geq 1} X^{\cdot}$ is compact. by induction on $n$, we get the assertion. Otherwise, Since we have $\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, A)}(X^{\cdot}, A^{(I)}) \cong \mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, A)}(X^{\cdot}, A)^{(I)}$, the canonical morphism $X^{\cdot} \to A^{(I)}$ factors through a direct summand $\mu : A^m \hookrightarrow A^{(I)}$ for some $m \in \mathbb{N}$. Then there is a homotopy morphism $h : X^1 \to A^{(I)}$ such that $1_{A^{(I)}} - \mu g = h d^0$ with some $g : A^{(I)} \to A^m$. Let $A^{(I)} = A^m \oplus A^{(J)}$ be the canonical decomposition, then $A^{(J)} \xrightarrow{d^0|_{A^{(J)}}} X^1 \xrightarrow{ph} A^{(J)} = 1_{A^{(J)}}$, where $p : A^{(I)} \to A^{(J)}$ is the canonical projection. Therefore $X^{\cdot} \cong \mathrm{M}^{\cdot}(1_{A^{(J)}})[-1] \oplus X'^{\cdot}$, where $X'^{\cdot} : A^m \to X'^1 \to \ldots \to X^n$ with $X'^1$ being a direct summand of $X^1$. Then we reduce the case of $X^0$ being a finitely generated free $A$-module. $\square$

**Lemma 16.3.** *For a complex $X^{\cdot} \in \mathsf{K}(\mathsf{Proj}\, A)$, the following hold.*

1. *$X^{\cdot}$ is a compact object in $\mathsf{K}(\mathsf{Mod}\, A)$.*
2. *There is a complex $P^{\cdot} \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ such that $X^{\cdot} \cong P^{\cdot}$ in $\mathsf{K}(\mathsf{Mod}\, A)$.*

*Proof.* $2 \Rightarrow 1$. We may assume $X^{\cdot} \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$. Let $\{Y_i^{\cdot}\}_{i \in I}$ be a collection of complexes of $\mathsf{K}(\mathsf{Mod}\, A)$. Since a finitely generated $A$-module is a compact object in $\mathsf{Mod}\, A$ (see Exercise 2.8), we have isomorphisms

$$\begin{aligned}\mathrm{Hom}^{\cdot}_A(X^{\cdot}, \textstyle\coprod_{i \in I} Y_i^{\cdot}) &\cong \mathrm{Tot}\, \mathrm{Hom}^{\cdot\cdot}_A(X^{\cdot}, \textstyle\coprod_{i \in I} Y_i^{\cdot}) \\ &\cong \textstyle\coprod_{i \in I} \mathrm{Tot}\, \mathrm{Hom}^{\cdot\cdot}_A(X^{\cdot}, Y_i^{\cdot}) \\ &\cong \textstyle\coprod_{i \in I} \mathrm{Hom}^{\cdot}_A(X^{\cdot}, Y_i^{\cdot}).\end{aligned}$$

By taking cohomology, we have an isomorphism

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, A)}(X^{\cdot}, \textstyle\coprod_{i \in I} Y_i^{\cdot}) \cong \textstyle\coprod_{i \in I} \mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, A)}(X^{\cdot}, Y_i^{\cdot}).$$

$1 \Rightarrow 2$. Since $\mathrm{C}^{\cdot}(X^{\cdot}) = \bigoplus_{i \in \mathbb{Z}} \mathrm{C}^i(X^{\cdot})[-i]$, we have isomorphisms in $\mathfrak{Ab}$

$$\begin{aligned}\textstyle\coprod_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{K}(\mathsf{Proj}\, A)}(X^{\cdot}, \mathrm{C}^i(X^{\cdot})[-i]) &\cong \mathrm{Hom}_{\mathsf{K}(\mathsf{Proj}\, A)}(X^{\cdot}, \textstyle\bigoplus_{i \in \mathbb{Z}} \mathrm{C}^i(X^{\cdot})[-i]) \\ &\cong \textstyle\prod_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{K}(\mathsf{Proj}\, A)}(X^{\cdot}, \mathrm{C}^i(X^{\cdot})[-i]).\end{aligned}$$

Then it is easy to see $\mathrm{Hom}_{\mathsf{K}(\mathsf{Proj}\,A)}(X^\bullet, \mathrm{C}^i(X^\bullet)[-i]) = 0$ for all but finitely many $i \in \mathbb{Z}$. Therefore for all but finitely many $i \in \mathbb{Z}$, we have exact sequences

$$\mathrm{Hom}_A(X^{i+1}, C^i(X^\bullet)) \to \mathrm{Hom}_A(C^i(X^\bullet), C^i(X^\bullet)) \to O.$$

This means that the canonical morphisms $C^i(X^\bullet) \to X^{i+1}$ are split monomorphisms. Then there are $m \le n$ such that $X^\bullet \cong \sigma'_{\ge m}\sigma_{\le n}X^\bullet$. $\sigma'_{\ge m}\sigma_{\le n}X^\bullet \in \mathsf{K}(\mathsf{Proj}\,A)$. Then we may assume $X^\bullet : X^0 \to X^1 \to \ldots \to X^n$ with $\mathrm{H}^0(X^\bullet) \ne 0$, $\mathrm{H}^n(X^\bullet) \ne 0$. By the proof of Lemma 16.2, we get the statement. □

**Proposition 16.4.** *Let $X^\bullet$ be a complex of $\mathsf{D}^*(\mathsf{Mod}\,A)$, where $* =$ nothing, $-$. Then the following are equivalent.*

1. *$X^\bullet$ is a perfect complex.*
2. *$X^\bullet$ is a compact object in $\mathsf{D}^*(\mathsf{Mod}\,A)$.*

*Proof.* Since $\mathsf{K}^{\mathrm{s}}(\mathsf{Proj}\,A) \overset{t}{\cong} \mathsf{D}(\mathsf{Mod}\,A)$, it is trivial by Lemma 16.3. □

**Lemma 16.5.** *Let $T^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ with $\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\,A)}(T^\bullet, T^\bullet[i]) = 0$ for $i \ne 0$, and $B = \mathrm{End}_{\mathsf{K}(\mathsf{Mod}\,A)}(T)$. Then there exists a fully faithful $\partial$-functor $F : \mathsf{K}^-(\mathsf{Proj}\,B) \to \mathsf{K}^-(\mathsf{Proj}\,A)$ such that*

1. *$FB \cong T^\bullet$.*
2. *$F$ preserves coproducts.*
3. *$F$ has a right adjoint $G : \mathsf{K}^-(\mathsf{Proj}\,A) \to \mathsf{K}^-(\mathsf{Proj}\,B)$.*

*Skip.* This lemma is important. But the proof is out of the methods of derived categories. □

**Lemma 16.6.** *If $T^\bullet$ satisfies the condition (G), then $F : \mathsf{K}^-(\mathsf{Proj}\,B) \to \mathsf{K}^-(\mathsf{Proj}\,A)$ is an equivalence.*

(G) *For $X^\bullet \in \mathsf{K}^-(\mathsf{Proj}\,A)$, $X^\bullet = O$ whenever $\mathrm{Hom}_{\mathsf{K}^-(\mathsf{Proj}\,A)}(T^\bullet, X^\bullet[i]) = 0$ for all $i$.*

*Proof.* Let $X^\bullet \in \mathsf{K}^-(\mathsf{Proj}\,A)$ such that $GX^\bullet = O$. Then $\mathrm{Hom}_{\mathsf{K}^-(\mathsf{Proj}\,A)}(T^\bullet, X^\bullet[i])$ $\cong \mathrm{Hom}_{\mathsf{K}^-(\mathsf{Proj}\,B)}(B, GX^\bullet[i]) = 0$ for all $i$. Therefore $\mathrm{Ker}\,G = \{O\}$. By the left version of Proposition 9.13, $G$ and $F$ are equivalences. □

**Definition 16.7.** Let $\mathcal{C}$ be a triangulated category. A subcategory $\mathcal{B}$ of $\mathcal{C}$ is said to *generates* $\mathcal{C}$ as a triangulated category if $\mathcal{C}$ is the smallest triangulated full subcategory which is closed under isomorphisms and contains $\mathcal{B}$.

**Remark 16.8.** Let $\mathcal{C}$ be a triangulated category. For a subcategory $\mathcal{B}$ of $\mathcal{C}$, we can construct the smallest triangulated full subcategory $\mathcal{EB}$ which is closed under isomorphisms and contains $\mathcal{B}$ as follows.

Let $\mathcal{E}^0\mathcal{B} = \mathcal{B}$. For $n > 0$, let $\mathcal{E}^n\mathcal{B}$ be the full subcategory of $\mathcal{C}$ consisting of objects $X$ there exist $U, V \in \mathcal{E}^{n-1}\mathcal{B}$ satisfying that either of $(X, U, V, *, *, *)$ or $(U, V, X, *, *, *)$ is a triangle in $\mathcal{C}$. Then it is easy to see that $\mathcal{EB} = \bigcup_{n\ge 0}\mathcal{E}^n\mathcal{B}$ is the smallest triangulated full subcategory which is closed under isomorphisms and contains $\mathcal{B}$

**Theorem 16.9.** *Let $T^\bullet$ be a complex of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ such that*

(a) $\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\,A)}(T^\bullet, T^\bullet[i]) = 0$ *for $i \ne 0$,*
(b) $\mathsf{add}\,T^\bullet_A$ *generates $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$.*

*Then $F : \mathsf{K}^-(\mathsf{Proj}\,B) \to \mathsf{K}^-(\mathsf{Proj}\,A)$ is an equivalence.*

*Proof.* It suffices to show that $T^\cdot$ satisfies the condition of Lemma 16.6. Since $\mathsf{add}\, T^\cdot_A$ generates $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$, if $\mathrm{Hom}_{\mathsf{K}^-(\mathsf{Proj}\, A)}(T^\cdot, X^\cdot[i]) = 0$ for all $i$, then $\mathrm{Hom}_{\mathsf{K}^-(\mathsf{Proj}\, A)}(A, X^\cdot[i]) = 0$ for all $i$. Thus $X^\cdot = O$. □

**Lemma 16.10.** *For $X^\cdot \in \mathsf{D}^-(\mathsf{Mod}\, A)$, the following are equivalent.*

1. $X^\cdot \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$.
2. *For any $Y^\cdot \in \mathsf{D}^-(\mathsf{Mod}\, A)$, there is $n$ such that $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A}(Y^\cdot, X^\cdot[i]) = 0$ for all $i < n$.*

*Proof.* 1 ⇒ 2. We may assume $X^\cdot \in \mathsf{C}^{\mathrm{b}}(\mathsf{Mod}\, A)$, $Y^\cdot \in \mathsf{K}^-(\mathsf{Proj}\, A)$. Then $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(Y^\cdot, X^\cdot[i]) \cong \mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, A)}(Y^\cdot, X^\cdot[i])$.

2 ⇒ 1. Since $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{b}})}(A, X^\cdot[i]) \cong \mathrm{H}^i(X^\cdot)$, it is easy. □

For an additive category $\mathcal{B}$ and $m \leq n$, we write $\mathsf{K}^{[m,n]}(\mathcal{B})$ for the full subcategory of $\mathsf{K}(\mathcal{B})$ consisting of complexes $X^\cdot$ with $X^i = O$ for $i < m$, $n < i$.

**Lemma 16.11.** *For $X^\cdot \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$, the following are equivalent.*

1. *$X^\cdot$ is isomorphic to an object of $\mathsf{K}^{\mathrm{b}}(\mathsf{Proj}\, A)$.*
2. *For any $Y^\cdot \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$, there is $n$ such that $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A}(X^\cdot, Y^\cdot[i]) = 0$ for all $i > n$.*

*Proof.* 1 ⇒ 2. It is trivial.

2 ⇒ 1. We may assume $X^\cdot \in \mathsf{K}^-(\mathsf{Proj}\, A)$. Let $M = \prod_{i \in \mathbb{Z}} \mathrm{C}^i(X^\cdot)$. By the same reason as the proof of Lemma 16.3, $\mathrm{Hom}_{\mathsf{K}^-(\mathsf{Mod}\, A)}(X^\cdot, M[i]) = 0$ for all $i > n$ if and only if $X^\cdot$ is isomorphic to an object in $\mathsf{K}^{[-n,\infty)}(\mathsf{Proj}\, A)$. □

**Theorem 16.12.** *Let $A$, $B$ be rings. The following are equivalent.*

1. $\mathsf{D}^-(\mathsf{Mod}\, A) \overset{t}{\cong} \mathsf{D}^-(\mathsf{Mod}\, B)$.
2. $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A) \overset{t}{\cong} \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B)$.
3. $\mathsf{K}^{\mathrm{b}}(\mathsf{Proj}\, A) \overset{t}{\cong} \mathsf{K}^{\mathrm{b}}(\mathsf{Proj}\, B)$.
4. $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A) \overset{t}{\cong} \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, B)$.
5. *There exists $T^\cdot \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ with $B \cong \mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)}(T^\cdot)$ such that*
   *(a) $\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, A)}(T^\cdot, T^\cdot[i]) = 0$ for $i \neq 0$,*
   *(b) $\mathsf{add}\, T^\cdot_A$ generates $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$.*
6. *There exists $T^\cdot \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ with $B \cong \mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)}(T^\cdot)$ such that*
   *(a) $\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, A)}(T^\cdot, T^\cdot[i]) = 0$ for $i \neq 0$,*
   *(b) For $X^\cdot \in \mathsf{K}^-(\mathsf{Proj}\, A)$, $X^\cdot = O$ whenever $\mathrm{Hom}_{\mathsf{K}^-(\mathsf{Proj}\, A)}(T^\cdot, X^\cdot[i]) = 0$ for all $i$.*

*Proof.* By Theorem 16.9, Lemmas 16.6, 16.10, 16.11 and 16.2. □

**Remark 16.13.** Since the functors $\mathrm{Hom}_A(-, A) : \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A) \to \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A^{\mathrm{op}})$ and $\mathrm{Hom}_A(-, A) : \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A^{\mathrm{op}}) \to \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ induce a duality between them, the condition 5 of Theorem 16.12 induces the left version of the condition 5. Therefore, $\mathsf{D}^-(\mathsf{Mod}\, A) \overset{t}{\cong} \mathsf{D}^-(\mathsf{Mod}\, B)$ if and only if $\mathsf{D}^-(\mathsf{Mod}\, A^{\mathrm{op}}) \overset{t}{\cong} \mathsf{D}^-(\mathsf{Mod}\, B^{\mathrm{op}})$.

**Definition 16.14.** A complex $T^\cdot_A \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ is called a *tilting complex* for $A$ provided that

1. $\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, A)}(T^\cdot, T^\cdot[i]) = 0$ for $i \neq 0$.

2. $\mathsf{add}\, T^{\boldsymbol{\cdot}}_A$ generates $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$.

We say that $B$ is *derived equivalent* to $A$ if there is a tilting complex $T^{\boldsymbol{\cdot}}_A$ such that $B \cong \mathrm{End}_{\mathsf{K}(\mathsf{Mod}\, A)}(T^{\boldsymbol{\cdot}})$.

**Lemma 16.15.** *Let* $\mathcal{U}$ *be a collection of objects* $\mathsf{K}^{[r,s]}(\mathsf{proj}\, A)$ *for some* $r \leq s$. *Consider a sequence of triangles*

$$\begin{aligned} U^{\boldsymbol{\cdot}}_1 \to U^{\boldsymbol{\cdot}}_0 \to X^{\boldsymbol{\cdot}}_1 \to \\ U^{\boldsymbol{\cdot}}_2[1] \to X^{\boldsymbol{\cdot}}_1 \to X^{\boldsymbol{\cdot}}_2 \to \\ \cdots \\ U^{\boldsymbol{\cdot}}_n[n-1] \to X^{\boldsymbol{\cdot}}_{n-1} \to X^{\boldsymbol{\cdot}}_n \to \\ \cdots, \end{aligned}$$

*with all* $U^{\boldsymbol{\cdot}}_i \in \mathcal{U}$. *Then* $\underrightarrow{\mathrm{hlim}}\, X^{\boldsymbol{\cdot}}_n \in \tilde{\mathsf{K}}^{-}(\mathsf{proj}\, A)$.

**Proposition 16.16.** *For rings A, B, the following are equivalent.*

1. *B is derived equivalent to A.*
2. $\mathsf{K}^{-}(\mathsf{proj}\, A) \overset{t}{\cong} \mathsf{K}^{-}(\mathsf{proj}\, B)$.

*In this case, we have* $\mathsf{K}^{-,\mathrm{b}}(\mathsf{proj}\, A) \overset{t}{\cong} \mathsf{K}^{-,\mathrm{b}}(\mathsf{proj}\, B)$.

*Proof.* $1 \Rightarrow 2$. According to Theorem 16.12, we have $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A) \overset{t}{\cong} \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, B)$. By Lemma 16.15, it is easy to see that $\mathsf{K}^{-}(\mathsf{proj}\, A) \overset{t}{\cong} \mathsf{K}^{-}(\mathsf{proj}\, B)$

$2 \Rightarrow 1$. Let $X^{\boldsymbol{\cdot}} \in \mathsf{K}^{-}(\mathsf{proj}\, A)$. Since $\bigoplus_{n\in\mathbb{N}} \tau_{\leq -n} X^{\boldsymbol{\cdot}}$ exists in $\mathsf{K}^{-}(\mathsf{proj}\, A)$, if $X^{\boldsymbol{\cdot}}$ is a compact object in $\mathsf{K}^{-}(\mathsf{proj}\, A)$, then

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, A)}(X^{\boldsymbol{\cdot}}, \tau_{\leq -n} X^{\boldsymbol{\cdot}}) = 0$$

for all but finitely many $n$. If $\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, A)}(X^{\boldsymbol{\cdot}}, \tau_{\leq -n} X^{\boldsymbol{\cdot}}) = 0$, then by Proposition 4.8, $X^{\boldsymbol{\cdot}} \cong \tau_{\geq -n+1} X^{\boldsymbol{\cdot}} \oplus \tau_{\leq -n} X^{\boldsymbol{\cdot}}[-1]$. According to Proposition 10.23 $X^{\boldsymbol{\cdot}} \in \tilde{\mathsf{K}}^{\mathrm{b}}(\mathsf{proj}\, A)$. As a consequence, $X^{\boldsymbol{\cdot}}$ is a compact object in $\mathsf{K}^{-}(\mathsf{proj}\, A)$ if and only if $X^{\boldsymbol{\cdot}}$ is isomorphic to an object in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$. Since compactness is a categorical property, we have $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A) \overset{t}{\cong} \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, B)$. By Theorem 16.12, we get the statement.

The last assertion is trivial by Lemma 16.10. □

**Lemma 16.17.** *For* $P^{\boldsymbol{\cdot}} \in \mathsf{C}^{-,\mathrm{b}}(\mathsf{Proj}\, A)$, *we have isomorphisms in* $\mathsf{K}^{-,\mathrm{b}}(\mathsf{Proj}\, A)$

$$\begin{aligned} \underrightarrow{\mathrm{hlim}}\, \tau_{\geq -i} P^{\boldsymbol{\cdot}} &\cong \underrightarrow{\lim}_{\mathsf{C}^{-,\mathrm{b}}(\mathsf{Proj}\, A)} \tau_{\geq -i} P^{\boldsymbol{\cdot}} \\ &\cong \underrightarrow{\lim}_{\mathsf{K}^{-,\mathrm{b}}(\mathsf{Proj}\, A)} \tau_{\geq -i} P^{\boldsymbol{\cdot}}. \end{aligned}$$

*Proof.* According to Proposition 11.7, we have the first isomorphism. For $Y^{\boldsymbol{\cdot}} \in \mathsf{C}^{-,\mathrm{b}}(\mathsf{Proj}\, A)$, there is $n \in \mathbb{Z}$ such that $\mathrm{H}^i(Y^{\boldsymbol{\cdot}}) = 0$ for all $i \leq n$. Applying $\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, A)}(-, Y^{\boldsymbol{\cdot}}[j])$ to a triangle $\tau_{\geq -i+1} P^{\boldsymbol{\cdot}} \to \tau_{\geq -i} P^{\boldsymbol{\cdot}} \to P^{-i}[i] \to \tau_{\geq -i+1} P^{\boldsymbol{\cdot}}[1]$, we have an exact sequence

$$\begin{aligned} &\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, A)}(P^{-i}[i], Y^{\boldsymbol{\cdot}}[j]) \to \mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, A)}(\tau_{\geq -i} P^{\boldsymbol{\cdot}}, Y^{\boldsymbol{\cdot}}[j]) \xrightarrow{\mu_i} \\ &\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, A)}(\tau_{\geq -i+1} P^{\boldsymbol{\cdot}}, Y^{\boldsymbol{\cdot}}[j]) \to \mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, A)}(P^{-i}[i], Y^{\boldsymbol{\cdot}}[j+1]). \end{aligned}$$

By Exercise 6.22, we have

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, A)}(P^{-i}, Y^{\boldsymbol{\cdot}}[j-i+1]) \cong \mathrm{Hom}_A(P^{-i}, \mathrm{H}^{j-i+1}(Y^{\boldsymbol{\cdot}})) = 0$$

for $i \geq j-n+1$, and then $\mu_i$ are epic for $i \geq j-n+1$. By Exercise 11.5, we have an exact sequence

$$0 \to \mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\,A)}(\varinjlim{}_{\mathsf{C}^{-,\mathrm{b}}(\mathsf{Proj}\,A)}\tau_{\geq -i}P^{\bullet}, Y^{\bullet}) \to \prod_i \mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\,A)}(\tau_{\geq -i}P^{\bullet}, Y^{\bullet}) \to \prod_i \mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\,A)}(\tau_{\geq -i}P^{\bullet}, Y^{\bullet}) \to 0.$$

Hence

$$\varinjlim{}_{\mathsf{C}^{-,\mathrm{b}}(\mathsf{Proj}\,A)}\tau_{\geq -i}P^{\bullet} \cong \varinjlim{}_{\mathsf{K}^{-,\mathrm{b}}(\mathsf{Proj}\,A)}\tau_{\geq -i}P^{\bullet}.$$

□

**Proposition 16.18.** *Let $A$, $B$ be coherent rings. The following are equivalent.*

1. *$B$ is derived equivalent to $A$.*
2. $\mathsf{D}^{-}(\mathsf{mod}\,A) \overset{t}{\cong} \mathsf{D}^{-}(\mathsf{mod}\,B)$.
3. $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A) \overset{t}{\cong} \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,B)$.

*Proof.* $1 \Rightarrow 2, 3$. By Proposition 16.16.

$3 \Rightarrow 1$. By Corollary 10.13, $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)$ and $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,B)$ are triangle equivalent to $\mathsf{K}^{-,\mathrm{b}}(\mathsf{proj}\,A)$ and $\mathsf{K}^{-,\mathrm{b}}(\mathsf{proj}\,B)$, respectively. Then we have an equivalence $F : \mathsf{K}^{-,\mathrm{b}}(\mathsf{proj}\,A) \to \mathsf{K}^{-,\mathrm{b}}(\mathsf{proj}\,B)$ and its quasi-inverse $G : \mathsf{K}^{-,\mathrm{b}}(\mathsf{proj}\,B) \to \mathsf{K}^{-,\mathrm{b}}(\mathsf{proj}\,A)$. We may assume that $G(B) = Q^{\bullet} : \ldots \to Q^{-1} \to Q^{0}$ and $F(A) = P^{\bullet} : \ldots \to P^{n-1} \to P^{n}$. By Proposition 11.7, $\tau_{\leq -1}P^{\bullet} \cong \varinjlim \tau_{\geq -i}\tau_{\leq -1}P^{\bullet}$. Since $G$ is an equivalence, by Lemma 16.17 $G\tau_{\leq -1}P^{\bullet} \cong \varinjlim{}_{\mathsf{K}^{-,\mathrm{b}}(\mathsf{Proj}\,A)}G\tau_{\geq -i}\tau_{\leq -1}P^{\bullet}$ in $\mathsf{K}^{-,\mathrm{b}}(\mathsf{proj}\,B)$. Since $G\tau_{\leq -1}P^{\bullet} \in \mathsf{K}^{-,\mathrm{b}}(\mathsf{proj}\,B)$, there exists $k \in \mathbb{Z}$ such that $\mathrm{H}^{k}(G\tau_{\leq -1}P^{\bullet}) \neq 0$ and $\mathrm{H}^{j}(G\tau_{\leq -1}P^{\bullet}) = 0$ for all $j > k$. Let $C^{\bullet}_{k} = \mathrm{C}^{k}(G\tau_{\leq -1}P^{\bullet})$, then $C^{\bullet}_{k} = \mathrm{C}^{k}(G\tau_{\leq -1}P^{\bullet}) \in \mathsf{K}^{-,\mathrm{b}}(\mathsf{proj}\,A)$ and $\mathrm{Hom}_{\mathsf{K}(\mathsf{proj}\,A)}(G\tau_{\leq -1}P^{\bullet}, C^{\bullet}_{k}[-k]) \neq 0$, because $A$ is coherent. Therefore, there is $m \geq 1$ such that $\mathrm{Hom}_{\mathsf{K}(\mathsf{proj}\,A)}(G\tau_{\geq -m}\tau_{\leq -1}P^{\bullet}, C^{\bullet}_{k}[-k]) \neq 0$. Then we have $m < 0$, because $Q^{\bullet} \in \mathsf{K}^{[-\infty,0]}(\mathsf{proj}\,A)$. Since

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{proj}\,A)}(P^{\bullet}, \tau_{\leq -1}P^{\bullet}) \cong \mathrm{Hom}_{\mathsf{K}(\mathsf{proj}\,A)}(A, G\tau_{\leq -1}P^{\bullet}) \cong \mathrm{H}^{0}(G\tau_{\leq -1}P^{\bullet}) = 0,$$

by Proposition 4.8 $\tau_{\geq 0}P^{\bullet} \cong P^{\bullet} \oplus \tau_{\leq -1}P^{\bullet}[-1]$. According to Proposition 10.23, $P^{\bullet} \in \tilde{\mathsf{K}}^{\mathrm{b}}(\mathsf{proj}\,A)$, and hence $P^{\bullet}_{A}$ is a tilting complex. □

## 17. Two-sided Tilting Complexes

17.1. **The Case of Flat $k$-algebras.** Throughout this subsection, $k$ is a commutative ring, $A, B, C$ are $k$-algebras which are $k$-flat modules. See Propositions 15.7, 15.8.

**Theorem 17.1.** *Let $A_i$ be an $k$-algebra with a tilting complex $T^{\bullet}_{i}$ whose endomorphism is isomorphic to $B_i$ ($i = 1, 2$). Then $T^{\bullet}_{1} \overset{\bullet}{\otimes}_{k} T^{\bullet}_{2}$ is a tilting complex for $A_1 \otimes_k A_2$ whose endomorphism is isomorphic to $B_1 \otimes_k B_2$.*

*Proof.* Since $T^{j}_{i}$ is $A_i$-projective, by Proposition 15.1 we have isomorphisms for all $i, j, k, l$

$$\begin{aligned}\mathrm{Hom}_{A_1\otimes_k A_2}(T^{i}_{1}\otimes_k T^{j}_{2}, T^{k}_{1}\otimes_k T^{l}_{2}) &\cong \mathrm{Hom}_{A_1}(T^{i}_{1}, \mathrm{Hom}_{A_1}(T^{j}_{2}, T^{k}_{1}\otimes_k T^{l}_{2}))\\ &\cong \mathrm{Hom}_{A_1}(T^{i}_{1}, A_1)\otimes_{A_1^{\mathrm{op}}} T^{k}_{1}\otimes_k T^{l}_{2}\otimes_{A_2} \mathrm{Hom}_{A_2}(T^{j}_{2}, A_2)\\ &\cong \mathrm{Hom}_{A_1}(T^{i}_{1}, T^{k}_{1})\otimes_k \mathrm{Hom}_{A_2}(T^{j}_{2}, T^{l}_{2}).\end{aligned}$$

This induces an isomorphism between quadruple complexes. Since $T_i^{\cdot}$ are bounded complexes, we have an isomorphism between complexes

$$\mathrm{Hom}^{\cdot}_{A_1\otimes_k A_2}(T_1^{\cdot} \dot{\otimes}_k T_2^{\cdot}, T_1^{\cdot} \dot{\otimes}_k T_2^{\cdot}) \xrightarrow{\sim} \mathrm{Hom}^{\cdot}_{A_1}(T_1^{\cdot}, T_1^{\cdot}) \dot{\otimes}_k \mathrm{Hom}^{\cdot}_{A_2}(T_2^{\cdot}, T_2^{\cdot}).$$

Since $A_i$, $B_i$ are $k$-flat, we have isomorphisms in $\mathsf{D}(\mathsf{Mod}\, k)$

$$\begin{aligned}\mathrm{Hom}^{\cdot}_{A_1}(T_1^{\cdot}, T_1^{\cdot}) \dot{\otimes}_k \mathrm{Hom}^{\cdot}_{A_2}(T_2^{\cdot}, T_2^{\cdot}) &\cong \mathrm{Hom}^{\cdot}_{A_1}(T_1^{\cdot}, T_1^{\cdot}) \dot{\otimes}_k^{\boldsymbol{L}} \mathrm{Hom}^{\cdot}_{A_2}(T_2^{\cdot}, T_2^{\cdot}) \\ &\cong \mathrm{End}_{\mathsf{K}(\mathsf{Mod}\, A_1)}(T_1^{\cdot}) \dot{\otimes}_k^{\boldsymbol{L}} \mathrm{End}_{\mathsf{K}(\mathsf{Mod}\, A_2)}(T_2^{\cdot}) \\ &\cong B_1 \dot{\otimes}_k^{\boldsymbol{L}} B_2 \\ &\cong B_1 \otimes_k B_2.\end{aligned}$$

Thus $\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, A_1\otimes_k A_2)}(T_1^{\cdot} \dot{\otimes}_k T_2^{\cdot}, T_1^{\cdot} \dot{\otimes}_k T_2^{\cdot}[i]) = 0$ for $i \neq 0$. It is easy to see that $\mathrm{End}_{\mathsf{K}(\mathsf{Mod}\, A_1\otimes_k A_2)}(T_1^{\cdot} \dot{\otimes}_k T_2^{\cdot}) \cong B_1 \otimes_k B_2$ as $k$-algebras.

Let $F = A_1\otimes_k - : \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A_2) \to \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A_1\otimes_k A_2)$. Since $\mathsf{add}\, T_2^{\cdot}$ generates $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A_2)$, $\mathsf{add}\, A_1\otimes_k T_2^{\cdot}$ generates $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A_1\otimes_k A_2)$. Let $G = -\dot{\otimes}_k T_2^{\cdot} : \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A_1) \to \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A_1\otimes_k A_2)$. Since $\mathsf{add}\, T_1^{\cdot}$ generates $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A_1)$, the triangulated full sub category generated by $\mathsf{add}\, T_1 \otimes_k T_2^{\cdot}$, contains $\mathsf{add}\, A_1\otimes_k T_2^{\cdot}$. Therefore $\mathsf{add}\, T_1\otimes_k T_2^{\cdot}$ generates $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A_1\otimes_k A_2)$. □

**Proposition 17.2.** *Let $A$ be a $k$-algebra which is derived equivalent to a $k$-algebra $B$. Then we have isomorphisms*

$$\begin{aligned}\mathrm{HH}_k^{\cdot}(A, A) &= \mathrm{Ext}^{\cdot}_{A^{\mathrm{op}}\otimes_k A}(A, A) \\ &\cong \mathrm{Ext}^{\cdot}_{B^{\mathrm{op}}\otimes_k B}(B, B) \\ &= \mathrm{HH}_k^{\cdot}(B, B).\end{aligned}$$

*Here* $\mathrm{HH}_k^{\cdot}$ *means Hochschild homology.*

*Proof.* Let $T^{\cdot}$ be a tilting complex for $A$ whose endomorphism ring is isomorphic to $B$. By Theorem 17.1, $T^{\cdot}\otimes_k T^{\cdot *}$ is a tilting complex for $A^{\mathrm{op}}\otimes_k A$ whose endomorphism ring is isomorphic to $B^{\mathrm{op}}\otimes_k B$, where $T^{\cdot *} = \mathrm{Hom}_A(T^{\cdot}, A)$ According to Lemma 16.5, there is a equivalence $F : \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B^{\mathrm{op}}\otimes_k B) \to \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{op}}\otimes_k A)$ which sends $B^{\mathrm{op}}\otimes_k B$ to $T^{\cdot}\otimes_k T^{\cdot *}$. Let $X^{\cdot} \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B^{\mathrm{op}}\otimes_k B)$ such that $FX^{\cdot} \cong A$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{op}}\otimes_k A)$. Then we have

$$\begin{aligned}\mathrm{H}^n(X^{\cdot}) &\cong \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B^{\mathrm{op}}\otimes_k B)}(B^{\mathrm{op}}\otimes_k B, X^{\cdot}[n]) \\ &\cong \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{op}}\otimes_k A)}(T^{\cdot}\otimes_k T^{\cdot *}, A[n]) \\ &\cong \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)}(T^{\cdot}, T^{\cdot **}[n]) \\ &\cong \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)}(T^{\cdot}, T^{\cdot}[n]).\end{aligned}$$

Hence $X^{\cdot} \cong B$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B^{\mathrm{op}}\otimes_k B)$. □

**Definition 17.3.** Let $A$ be a $k$-algebra which is derived equivalent to a $k$-algebra $B$, and $P^{\cdot}$ a tilting complex for $A$ whose endomorphism ring is isomorphic to $B$. Then we have a triangle equivalence $F : \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B) \to \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$ which sends $B$ to $P^{\cdot}$. We take a complex $Q^{\cdot}$ of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, B)$ which is isomorphic to $F^{-1}A$ in $\mathsf{K}^{\mathrm{b}}(\mathsf{Mod}\, B)$, and $P^{\cdot *} = \mathrm{Hom}_A(P^{\cdot}, A)$, $Q^{\cdot\, *} = \mathrm{Hom}_B(Q^{\cdot}, B)$. Then a tilting complex $B\otimes_k P^{\cdot}$ induces the triangle equivalence

$$\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B^{\mathrm{op}}\otimes_k B) \to \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B^{\mathrm{op}}\otimes_k A).$$

| $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{op}}\otimes_k A)$ | $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B^{\mathrm{op}}\otimes_k A)$ | $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{op}}\otimes_k B)$ | $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B^{\mathrm{op}}\otimes_k B)$ |
|---|---|---|---|
| $A$ | $T^{\cdot}$ | $T^{\vee\cdot}$ | $B$ |

Let $T^{\cdot}$ be the image of $B$ of $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B^{\mathrm{op}}\otimes_k B)$. Also, a tilting complex $A\otimes_k Q^{\cdot\star}$ induces the triangle equivalence

$$\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{op}}\otimes_k A) \to \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{op}}\otimes_k B).$$

Let $T^{\vee\cdot}$ be the image of $A$ of $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{op}}\otimes_k A)$.

**Proposition 17.4.** *The following hold.*

1. $Res_A T^{\cdot} \cong P^{\cdot}$ *in* $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$.
2. $Res_{B^{\mathrm{op}}} T^{\cdot} \cong Q^{\cdot\star}$ *in* $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B^{\mathrm{op}})$.
3. $Res_B T^{\vee\cdot} \cong Q^{\cdot}$ *in* $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B)$.
4. $Res_{A^{\mathrm{op}}} T^{\vee\cdot} \cong P^{\cdot *}$ *in* $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{op}})$.

*Proof.* 1. We have isomorphisms of functors $\mathsf{K}^{-}(\mathsf{proj}\, A) \to \mathsf{Mod}\, k$

$$\begin{aligned}\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(-, Res_A T^{\cdot}) &\cong \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, B^{\mathrm{op}}\otimes_k A)}(B^{\mathrm{op}}\otimes_k -, T^{\cdot})\\ &\cong \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{op}}\otimes_k A)}(P^{\cdot *}\otimes_k -, A)\\ &\cong \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(-, P^{\cdot **})\\ &\cong \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(-, P^{\cdot}).\end{aligned}$$

2. We have isomorphisms of functors $\mathsf{K}^{-}(\mathsf{proj}\, A) \to \mathsf{Mod}\, k$

$$\begin{aligned}\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, B^{\mathrm{op}})}(-, Res_{B^{\mathrm{op}}} T^{\cdot}) &\cong \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, B^{\mathrm{op}}\otimes_k A)}(-\otimes_k A, T^{\cdot})\\ &\cong \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, B^{\mathrm{op}}\otimes_k B)}(-\otimes_k Q^{\cdot}, B)\\ &\cong \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, B^{\mathrm{op}})}(-, Q^{\cdot\star}).\end{aligned}$$

3, 4. Similarly. □

**Lemma 17.5.** *There is an isomorphism* $\phi : P^{\cdot} \to Res_A T^{\cdot}$ *in* $\mathsf{D}(\mathsf{Mod}\, A)$ *such that* $\phi f = \lambda(f)\phi$ *for all* $f \in \mathrm{End}_{\mathsf{D}(\mathsf{Mod}\, A)}(P^{\cdot})$*, where* $\lambda : B \to \mathrm{End}_{\mathsf{D}(\mathsf{Mod}\, A)}(Res_A T^{\cdot})$ *is the left multiplication morphism.*

*Proof.* For $f \in \mathrm{End}_{\mathsf{D}(\mathsf{Mod}\, A)}(P^{\cdot})$, by the above isomorphisms, we have a commutative diagram

$$\begin{array}{ccccc}\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(-, Res_A T^{\cdot}) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, B^{\mathrm{op}}\otimes_k A)}(B^{\mathrm{op}}\otimes_k -, T^{\cdot}) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(-, P^{\cdot})\\ {\scriptstyle \mathrm{Hom}(-,\lambda(f))}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(f^{\mathrm{op}}\otimes -, A)} & & \downarrow{\scriptstyle \mathrm{Hom}(-,f)}\\ \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(-, Res_A T^{\cdot}) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, B^{\mathrm{op}}\otimes_k A)}(B^{\mathrm{op}}\otimes_k -, T^{\cdot}) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(-, P^{\cdot}).\end{array}$$

□

**Theorem 17.6.** *For* $* =$ *nothing,* $+, -, b$*, the* $\partial$*-functor*

$$- \overset{\bullet}{\otimes}{}^{\boldsymbol{L}*}_{B} T^{\cdot} : \mathsf{D}^{*}(\mathsf{Mod}\, B) \to \mathsf{D}^{*}(\mathsf{Mod}\, A)$$

*is an triangle equivalence, and its quasi-inverse is*

$$\boldsymbol{R}^{*}\,\mathrm{Hom}_A(T^{\cdot}, -) \cong - \overset{\bullet}{\otimes}{}^{\boldsymbol{L}*}_{A} T^{\vee\cdot} : \mathsf{D}^{*}(\mathsf{Mod}\, A) \to \mathsf{D}^{*}(\mathsf{Mod}\, B).$$

*Proof.* For a complex $X^\bullet \in \mathsf{D}^*(\mathsf{Mod}\, B)$, $X^\bullet \overset{\bullet}{\otimes}{}^{\boldsymbol{L}*}_B T^\bullet \cong X^\bullet \overset{\bullet}{\otimes}_B Q^{\bullet\star}$ in $\mathsf{D}(\mathsf{Mod}\, k)$. Then $-\overset{\bullet}{\otimes}{}^{\boldsymbol{L}*}_B T^\bullet$ is a way-out in both directions. Similarly, $\boldsymbol{R}^* \operatorname{Hom}_A(T^\bullet, -)$, $-\overset{\bullet}{\otimes}{}^{\boldsymbol{L}*}_A T^{\vee\bullet}$ are way-out in both directions. Therefore we have the above functors between the above derived categories. By Proposition 17.4, We have

$$\begin{aligned} Res_A T^\bullet \overset{\bullet}{\otimes}{}^{\boldsymbol{L}*}_A Res_{A^{\mathrm{op}}} T^{\vee\bullet} &\cong P^\bullet \overset{\bullet}{\otimes}{}^{\boldsymbol{L}*}_A P^{\bullet *} \\ &\cong \operatorname{End}_{\mathsf{D}^*(\mathsf{Mod}\, A)}(P^\bullet) \\ &= B \end{aligned}$$

By Lemma 17.5, we have $T^\bullet \overset{\bullet}{\otimes}{}^{\boldsymbol{L}*}_A T^{\vee\bullet} \cong B$ in $\mathsf{D}(\mathsf{Mod}\, B^{\mathrm{op}} \otimes_k B)$. Similarly we have $T^{\vee\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}*}_B T^\bullet \cong A$ in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{op}} \otimes_k A)$. Then $-\overset{\bullet}{\otimes}{}^{\boldsymbol{L}*}_B T^\bullet$ is an equivalence. By adjointness, we have $\boldsymbol{R}^* \operatorname{Hom}_A(T^\bullet, -) \cong -\overset{\bullet}{\otimes}{}^{\boldsymbol{L}*}_A T^{\vee\bullet}$. □

**Definition 17.7** (Biperfect Complex)**.** A complex $X^\bullet \in \mathsf{D}(\mathsf{Mod}\, A^{\mathrm{op}} \otimes_k B)$ is called a *biperfect complex* if $Res_A X^\bullet \in \mathsf{D}(\mathsf{Mod}\, A)_{\mathrm{perf}}$ and $Res_{B^{\mathrm{op}}} X^\bullet \in \mathsf{D}(\mathsf{Mod}\, B^{\mathrm{op}})_{\mathrm{perf}}$. We denote by $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{op}} \otimes_k B)_{\mathrm{biperf}}$ the triangulated full subcategory of $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{op}} \otimes_k B)$ consisting of biperfect complexes.

**Definition 17.8.** A bimodule complex ${}_B T^\bullet_A \in \mathsf{K}(\mathsf{Mod}\, B^{\mathrm{op}} \otimes_k A)$ is called a *two-sided tilting complex* provided that

1. ${}_B T^\bullet_A$ is a biperfect complex.
2. There exists a biperfect complex ${}_A T^{\vee\bullet}_B$ such that
   (a) ${}_B T^\bullet \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A T^{\vee\bullet}_B \cong B$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B^{\mathrm{op}} \otimes_k B)$,
   (b) ${}_A T^{\vee\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B T^\bullet_A \cong A$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{op}} \otimes_k A)$.

We call ${}_A T^{\vee\bullet}_B$ the *inverse* of ${}_B T^\bullet_A$.

$\boldsymbol{R}^* \operatorname{Hom}_A(T^\bullet, -) : \mathsf{D}^*(\mathsf{Mod}\, A) \to \mathsf{D}^*(\mathsf{Mod}\, B)$ is called a *standard equivalence*, where $* =$ nothing, $+, -, b$.

**Theorem 17.9.** *The following are equivalent.*

1. $\mathsf{D}(\mathsf{Mod}\, A) \overset{t}{\cong} \mathsf{D}(\mathsf{Mod}\, B)$.
2. $\mathsf{D}^+(\mathsf{Mod}\, A) \overset{t}{\cong} \mathsf{D}^+(\mathsf{Mod}\, B)$.
3. *$A$ is derived equivalent to $B$.*
4. *There exists a two-sided tilting complex ${}_B T^\bullet_A$.*

*Proof.* By Theorem 17.6 and the dual of Lemma 16.10. □

**Corollary 17.10.** *Let ${}_B T^\bullet_A$ and ${}_C S^\bullet_B$ be two-sided tilting complexes. Then ${}_C S^\bullet_B \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B T^\bullet_A$ is a two-sided tilting complex.*

17.2. **The Case of Projective $k$-algebras.** Throughout this subsection, $k$ is a commutative ring, $A, B, C$ are $k$-algebras which are $k$-projective modules. See Propositions 15.7, 15.8.

**Lemma 17.11.** *Let $X$, $P$ be $B$-$A$-bimodules such that $Res_A P$ is a finitely generated projective $A$-module. Then the following hold.*

1. *For a right $A$-module $M$ and a left $A$-module $N$, we have $B$-module morphisms*

$$M \otimes_A \operatorname{Hom}_A(X, A) \to \operatorname{Hom}_A(X, M) \ (m \otimes f \mapsto (x \mapsto m f(x))),$$

$$X \otimes_A N \to \operatorname{Hom}_A(\operatorname{Hom}_A(X, A), N) \ (x \otimes n \mapsto (f \mapsto f(x) n)).$$

2. *We have a functorial isomorphism of functors* $\mathsf{Mod}\,A \to \mathsf{Mod}\,B$
$$-\otimes_A \operatorname{Hom}_A(P, A) \cong \operatorname{Hom}_A(P, -).$$
3. *We have a functorial isomorphism of functors* $\mathsf{Mod}\,A^{\mathrm{op}} \to \mathsf{Mod}\,B^{\mathrm{op}}$
$$P\otimes_A - \cong \operatorname{Hom}_A(\operatorname{Hom}_A(P, A), -).$$

**Lemma 17.12.** *Let* $X^{\cdot} \in \mathsf{D}^*(\mathsf{Mod}\,B^{\mathrm{op}}\otimes_k A)$ *with* $Res_A X^{\cdot} \in \mathsf{D}^*(\mathsf{Mod}\,A)_{\mathrm{perf}}$, *where* $* =$ *nothing,* $+, -, b$. *Then the following hold.*

1. *We have a* $\partial$*-functorial isomorphism of* $\partial$*-functors* $\mathsf{D}^*(\mathsf{Mod}\,A^{\mathrm{op}}\otimes_k A) \to \mathsf{D}^*(\mathsf{Mod}\,A^{\mathrm{op}}\otimes_k B)$
$$-\dot{\otimes}_A^{\boldsymbol{L}*}\boldsymbol{R}\operatorname{Hom}^{\cdot}_A(X^{\cdot}, A) \cong \boldsymbol{R}^*\operatorname{Hom}^{\cdot}_A(X^{\cdot}, -).$$
2. *We have a* $\partial$*-functorial isomorphism of* $\partial$*-functors* $\mathsf{D}^*(\mathsf{Mod}\,A^{\mathrm{op}}\otimes_k A) \to \mathsf{D}^*(\mathsf{Mod}\,B^{\mathrm{op}}\otimes_k A)$
$$X^{\cdot}\dot{\otimes}_A^{\boldsymbol{L}*} - \cong \boldsymbol{R}^*\operatorname{Hom}^{\cdot}_A(\boldsymbol{R}\operatorname{Hom}^{\cdot}_A(X^{\cdot}, A), -).$$

*Proof.* 1. By Lemma 17.11, we have a $\partial$-functorial morphism of $\partial$-functors $\mathsf{D}^*(\mathsf{Mod}\,A^{\mathrm{op}}\otimes_k A) \to \mathsf{D}^*(\mathsf{Mod}\,A^{\mathrm{op}}\otimes_k B)$
$$\phi : -\dot{\otimes}_A^{\boldsymbol{L}}\boldsymbol{R}\operatorname{Hom}^{\cdot}_A(X^{\cdot}, A) \to \boldsymbol{R}\operatorname{Hom}^{\cdot}_A(X^{\cdot}, -).$$
Let $Q^{\cdot} \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ which has a quasi-isomorphism $Q^{\cdot} \to Res_A X^{\cdot}$. By Lemma 17.11, we have a $\partial$-functorial isomorphism of $\partial$-functors $\mathsf{D}^*(\mathsf{Mod}\,A) \to \mathsf{D}^*(\mathsf{Mod}\,k)$
$$\psi : -\dot{\otimes}_A \operatorname{Hom}^{\cdot}_A(Q^{\cdot}, A) \xrightarrow{\sim} \operatorname{Hom}^{\cdot}_A(Q^{\cdot}, -).$$
Since
$$Res_k\phi \cong \psi,$$
and $\mathrm{H}^{\cdot}(\psi)$ is an isomorphism, $\phi$ is a functorial isomorphism.

2. Similarly. ☐

**Corollary 17.13.** *Let* $T^{\cdot}$ *and* $T^{\vee\cdot}$ *be a two-sided tilting complex and its inverse. Then we have isomorphisms in* $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,B^{\mathrm{op}}\otimes_k A)$
$$\begin{aligned} T^{\vee\cdot} &\cong \boldsymbol{R}\operatorname{Hom}^{\cdot}_A(T^{\cdot}, A) \\ &\cong \boldsymbol{R}\operatorname{Hom}^{\cdot}_B(T^{\cdot}, B). \end{aligned}$$

**Theorem 17.14.** *For a bimodule complex* ${}_BT^{\cdot}_A$, *the following are equivalent.*

1. ${}_BT^{\cdot}_A$ *is a two-sided tilting complex.*
2. ${}_BT^{\cdot}_A$ *satisfies that*
   (a) ${}_BT^{\cdot}_A$ *is a biperfect complex,*
   (b) *The right multiplication morphism* $\rho_A : A \to \boldsymbol{R}\operatorname{Hom}^{\cdot}_B(T^{\cdot}, T^{\cdot})$ *is an isomorphism in* $\mathsf{D}(\mathsf{Mod}\,A^{\mathrm{op}}\otimes_k A)$,
   (c) *The left multiplication morphism* $\lambda_B : B \to \boldsymbol{R}\operatorname{Hom}^{\cdot}_A(T^{\cdot}, T^{\cdot})$ *is an isomorphism in* $\mathsf{D}(\mathsf{Mod}\,B^{\mathrm{op}}\otimes_k B)$.
3. ${}_BT^{\cdot}_A$ *satisfies that*
   (a) ${}_BT^{\cdot}_A$ *is a biperfect complex,*
   (b) $\operatorname{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,B^{\mathrm{op}})}(T^{\cdot}, T^{\cdot}[i]) = 0$ *for* $i \neq 0$,
   (c) $\operatorname{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,A)}(T^{\cdot}, T^{\cdot}[i]) = 0$ *for* $i \neq 0$,
   (d) *The right multiplication morphism* $\rho_A$ *induces a ring isomorphism* $A \to \operatorname{End}_{\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,B^{\mathrm{op}})}(T^{\cdot})^{\mathrm{op}}$,

*(e) The left multiplication morphism $\lambda_B$ induces a ring isomorphism $B \to \operatorname{End}_{\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,A)}(T^{\boldsymbol{\cdot}})$.*

*Proof.* 1 $\Rightarrow$ 3. By Corollary 17.13, Lemma 17.12, we have isomorphisms in $\mathsf{D}(\mathsf{Mod}\,B^{\mathrm{op}}\otimes_k B)$

$$\begin{aligned}\boldsymbol{R}\operatorname{Hom}^{\boldsymbol{\cdot}}_A(T^{\boldsymbol{\cdot}},T^{\boldsymbol{\cdot}}) &\cong {}_BT^{\boldsymbol{\cdot}} \dot{\otimes}{}^{\boldsymbol{L}}_A \boldsymbol{R}\operatorname{Hom}^{\boldsymbol{\cdot}}_A(T^{\boldsymbol{\cdot}},A)\\ &\cong {}_BT^{\boldsymbol{\cdot}} \dot{\otimes}{}^{\boldsymbol{L}}_A T^{\vee\boldsymbol{\cdot}}_B\\ &\cong B.\end{aligned}$$

And we have an isomorphism in $\mathsf{D}(\mathsf{Mod}\,A^{\mathrm{op}}\otimes_k A)$

$$\begin{aligned}\boldsymbol{R}\operatorname{Hom}^{\boldsymbol{\cdot}}_B(T^{\boldsymbol{\cdot}},T^{\boldsymbol{\cdot}}) &\cong \boldsymbol{R}\operatorname{Hom}^{\boldsymbol{\cdot}}_B(T^{\boldsymbol{\cdot}},B) \dot{\otimes}{}^{\boldsymbol{L}}_B T^{\boldsymbol{\cdot}}_A\\ &\cong {}_AT^{\vee\boldsymbol{\cdot}} \dot{\otimes}{}^{\boldsymbol{L}}_B T^{\boldsymbol{\cdot}}_A\\ &\cong A.\end{aligned}$$

Since $-\otimes^{\boldsymbol{L}}_B T^{\boldsymbol{\cdot}}_A$ is an equivalence, we have a commutative diagram

$$\begin{array}{ccc}\operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\,B)}(B_B,B_B) & \xrightarrow{\sim} & \operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\,A)}(B\otimes^{\boldsymbol{L}}_B T^{\boldsymbol{\cdot}}_A, B\otimes^{\boldsymbol{L}}_B T^{\boldsymbol{\cdot}}_A)\\ \uparrow & & \downarrow\\ B & \xrightarrow{\lambda_B} & \operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\,A)}(T^{\boldsymbol{\cdot}}_A,T^{\boldsymbol{\cdot}}_A)\end{array}$$

where all vertical arrows are isomorphisms. Then $\lambda_B$ is an isomorphism. Similarly, $\rho_A$ is an isomorphism.

3 $\Rightarrow$ 2. It is easy to see that we have a morphism $\lambda_A : A \to \boldsymbol{R}\operatorname{Hom}^{\boldsymbol{\cdot}}_B(T^{\boldsymbol{\cdot}},T^{\boldsymbol{\cdot}})$ in $\mathsf{D}(\mathsf{Mod}\,A^{\mathrm{op}}\otimes_k A)$. By taking cohomologies, we get the condition (b) of 2. Similarly, we get the condition (c) of 2.

2 $\Rightarrow$ 1. Let $T^{\vee\boldsymbol{\cdot}} = \boldsymbol{R}\operatorname{Hom}^{\boldsymbol{\cdot}}_A(T^{\boldsymbol{\cdot}},A)$, then we have isomorphisms

$$\begin{aligned}{}_BT^{\boldsymbol{\cdot}} \dot{\otimes}{}^{\boldsymbol{L}}_A T^{\vee\boldsymbol{\cdot}}_B &= {}_BT^{\boldsymbol{\cdot}} \dot{\otimes}{}^{\boldsymbol{L}}_A \boldsymbol{R}\operatorname{Hom}^{\boldsymbol{\cdot}}_A(T^{\boldsymbol{\cdot}},A)\\ &\cong \boldsymbol{R}\operatorname{Hom}^{\boldsymbol{\cdot}}_A(T^{\boldsymbol{\cdot}},T^{\boldsymbol{\cdot}})\\ &\cong B.\end{aligned}$$

By Proposition 15.5 2, we have an isomorphism

$$\begin{aligned}\boldsymbol{R}\operatorname{Hom}^{\boldsymbol{\cdot}}_A(T^{\boldsymbol{\cdot}},A) &\cong \boldsymbol{R}\operatorname{Hom}^{\boldsymbol{\cdot}}_A(T^{\boldsymbol{\cdot}},\boldsymbol{R}\operatorname{Hom}^{\boldsymbol{\cdot}}_B(T^{\boldsymbol{\cdot}},T^{\boldsymbol{\cdot}}))\\ &\cong \boldsymbol{R}\operatorname{Hom}^{\boldsymbol{\cdot}}_B(T^{\boldsymbol{\cdot}},\boldsymbol{R}\operatorname{Hom}^{\boldsymbol{\cdot}}_A(T^{\boldsymbol{\cdot}},T^{\boldsymbol{\cdot}}))\\ &\cong \boldsymbol{R}\operatorname{Hom}^{\boldsymbol{\cdot}}_B(T^{\boldsymbol{\cdot}},B).\end{aligned}$$

Then we have

$$\begin{aligned}{}_AT^{\vee\boldsymbol{\cdot}} \dot{\otimes}{}^{\boldsymbol{L}}_B T^{\boldsymbol{\cdot}}_A &\cong \boldsymbol{R}\operatorname{Hom}^{\boldsymbol{\cdot}}_B(T^{\boldsymbol{\cdot}},B) \dot{\otimes}{}^{\boldsymbol{L}}_B T^{\boldsymbol{\cdot}}_A\\ &\cong \boldsymbol{R}\operatorname{Hom}^{\boldsymbol{\cdot}}_B(T^{\boldsymbol{\cdot}},T^{\boldsymbol{\cdot}})\\ &\cong A.\end{aligned}$$

□

**Theorem 17.15.** *Let $(A_i, B_i)$ be derived equivalent $k$-algebras, $T^{\boldsymbol{\cdot}}_i$ two-sided tilting complexes in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,B_i^{\mathrm{op}}\otimes_k A_i)$ and their inverses $T^{\vee\boldsymbol{\cdot}}_i$ $(i = 0, 1, 2)$. Then we have the following commutative diagrams.*

1.

$$\begin{array}{ccc}
\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,A_0^{\mathrm{op}}\otimes_k A_1)\times \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,A_1^{\mathrm{op}}\otimes_k A_2) & \xrightarrow{-\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{A_1}-} & \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,A_0^{\mathrm{op}}\otimes_k A_2) \\
{\scriptstyle F_0\times F_1}\downarrow & & \downarrow{\scriptstyle F_2} \\
\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,B_0^{\mathrm{op}}\otimes_k B_1)\times \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,B_1^{\mathrm{op}}\otimes_k B_2) & \xrightarrow{-\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{B_1}-} & \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,B_0^{\mathrm{op}}\otimes_k B_2).
\end{array}$$

2.

$$\begin{array}{ccc}
\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,A_1^{\mathrm{op}}\otimes_k A_2)\times \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,A_0^{\mathrm{op}}\otimes_k A_2) & \xrightarrow{\boldsymbol{R}\mathrm{Hom}^{\bullet}_{A_2}(-,-)} & \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,A_0^{\mathrm{op}}\otimes_k A_1) \\
{\scriptstyle F_1\times F_2}\downarrow & & \downarrow{\scriptstyle F_0} \\
\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,B_1^{\mathrm{op}}\otimes_k B_2)\times \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,B_0^{\mathrm{op}}\otimes_k B_2) & \xrightarrow{\boldsymbol{R}\mathrm{Hom}^{\bullet}_{B_2}(-,-)} & \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,B_0^{\mathrm{op}}\otimes_k B_1).
\end{array}$$

*Here* $F_0 = T_0^{\bullet}\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{A_0}-\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{A_1}T_1^{\vee\bullet}$, $F_1 = T_1^{\bullet}\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{A_1}-\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{A_2}T_2^{\vee\bullet}$, $F_2 = T_0^{\bullet}\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{A_0}-\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{A_2}T_2^{\vee\bullet}$.

*Proof.* 1. We have isomorphisms

$$\begin{aligned}
&(T_0^{\bullet}\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{A_0}-\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{A_1}T_1^{\vee\bullet})\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{B_1}(T_1^{\bullet}\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{A_1}-\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{A_2}T_2^{\vee\bullet}) \\
&\cong (T_0^{\bullet}\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{A_0}-)\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{A_1}(T_1^{\vee\bullet}\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{B_1}T_1^{\bullet})\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{A_1}(-\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{A_2}T_2^{\vee\bullet}) \\
&\cong (T_0^{\bullet}\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{A_0}-)\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{A_1}(A_1)\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{A_1}(-\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{A_2}T_2^{\vee\bullet}) \\
&\cong (T_0^{\bullet}\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{A_0}-)\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{A_1}(-\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{A_2}T_2^{\vee\bullet}).
\end{aligned}$$

2. Let $X^{\bullet}\in\mathsf{Mod}\,A_0^{\mathrm{op}}\otimes_k A_2$, $Y^{\bullet}\in\mathsf{Mod}\,A_1^{\mathrm{op}}\otimes_k A_2$, $Z^{\bullet}\in\mathsf{Mod}\,A_0^{\mathrm{op}}\otimes_k A_2$. Since we have an adjoint isomorphism

$$\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,A_0^{\mathrm{op}}\otimes_k A_2)}(X^{\bullet}\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{A_1}Y^{\bullet},Z^{\bullet})\cong \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,A_0^{\mathrm{op}}\otimes_k A_1)}(X^{\bullet},\boldsymbol{R}\mathrm{Hom}^{\bullet}_{A_2}(Y^{\bullet},Z^{\bullet})),$$

we get the assertion by 1. □

17.3. **The Case of Finite Dimensional $k$-algebras.** Throughout this subsection, we assume $k$ is a field, and all algebra are finite dimensional $k$-algebras. We denote $D_k = \mathrm{Hom}_k(-,k)$.

**Definition 17.16** (Nakayama Functor). A triangle auto-equivalence $\nu_A = -\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{A}D_kA : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)\to\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)$ is called a *Nakayama functor*.

**Proposition 17.17.** *Let $(A,B)$ be derived equivalent $k$-algebras, $T^{\bullet}$ a two-sided tilting complex in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,B^{\mathrm{op}}\otimes_k A)$ and its inverse $T^{\vee\bullet}$. Then we have a commutative diagram*

$$\begin{array}{ccc}
\mathsf{D}^{-}(\mathsf{Mod}\,A) & \xrightarrow{\nu_A} & \mathsf{D}^{-}(\mathsf{Mod}\,A) \\
{\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\
\mathsf{D}^{-}(\mathsf{Mod}\,B) & \xrightarrow{\nu_B} & \mathsf{D}^{-}(\mathsf{Mod}\,B),
\end{array}$$

*where $F$ is a standard equivalence.*

*Proof.* By Proposition 17.2, the standard equivalence $G : \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,A^{\mathrm{op}}\otimes_k A)\to\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,B^{\mathrm{op}}\otimes_k B)$ sends $A$ to $B$.

By the case of $A_1 = A^{\mathrm{op}}\otimes_k A$, $B_1 = B^{\mathrm{op}}\otimes_k B$ and $A_0 = A_2 = B_0 = B_2 = k$ in Theorem 17.15 2, we have $GD_kA\cong D_kB$.

By the case of $A_1 = A_2 = A$, $B_1 = B_2 = B$ and $A_0 = B_0 = k$ in Theorem 17.15 1, we have $F\nu_A = \nu_B F$. □

**Corollary 17.18.** *Let $A$ be a finite dimensional $k$-algebra which is derived equivalent to $B$. If $A$ is a symmetric algebra, then $B$ is a symmetric algebra.*

**Lemma 17.19.** *Let $A$, $B$ be finite dimensional self-injective $k$-algebras, ${}_AP_B$ is $A^{\mathrm{op}}\otimes_k B$-projective, ${}_BV_A$ is $A$-projective and $B$-projective. The following hold.*

1. *$B^{\mathrm{op}}\otimes_k A$ is a self-injective algebra.*
2. *$\mathrm{Hom}_A(P, A)$ is $B^{\mathrm{op}}\otimes_k A$-projective.*
3. *$\mathrm{Hom}_A(V, A)$ is $A$-projective and $B$-projective.*
4. *${}_AP\otimes_B V_A$ is $A^{\mathrm{op}}\otimes_k A$-projective.*
5. *$X\otimes_A P_B$ is $B$-projective for any $X \in \mathsf{Mod}\, A$.*
6. *$Y\otimes_B V_A$ is $A$-projective for any $Y \in \mathsf{Proj}\, B$.*

*Proof.* By Propositions 15.1, 15.3. □

**Proposition 17.20.** *Let $A$ and $B$ be derived equivalent self-injective $k$-algebras. Then $\mathsf{K}(\mathsf{mod}\, A) \overset{t}{\cong} \mathsf{K}(\mathsf{mod}\, B)$, and there are bimodules ${}_AM_B$ and ${}_BN_A$ such that*

$$-\otimes_A M : \mathsf{mod}\, A \to \mathsf{mod}\, B \quad \textit{and} - \otimes_B N : \mathsf{mod}\, B \to \mathsf{mod}\, A$$

*induce an equivalence $\underline{\mathsf{mod}}A \cong \underline{\mathsf{mod}}B$.*

*Proof.* Let $T^{\bullet}$ be a two-sided tilting complex in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B^{\mathrm{op}}\otimes_k A)$ and $T^{\vee\bullet}$ its inverse. By taking a $B^{\mathrm{op}}\otimes_k A$-projective resolution of $T^{\bullet}$, and its shifting and truncation, we may assume $T^{\bullet}$ is isomorphic to

$$S^{\bullet} : S^{-n} \to S^{-n+1} \to \dots \to S^0$$

where $S^i$ are $B^{\mathrm{op}}\otimes_k A$-projective $(-n < i \leq 0)$, and $S^{-n}$ is $A$-projective and $B$-projective. Then $T^{\vee\bullet} \cong \mathrm{Hom}_A(S^{\bullet}, A)$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{op}}\otimes_k B)$, and we have

$$S^{\bullet} \dot{\otimes}_A \mathrm{Hom}_A(S^{\bullet}, A) \cong B \quad \text{in} \quad \mathsf{K}^{\mathrm{b}}(\mathsf{Mod}\, B^{\mathrm{op}}\otimes_k B),$$
$$\mathrm{Hom}_A(S^{\bullet}, A) \dot{\otimes}_B S^{\bullet} \cong A \quad \text{in} \quad \mathsf{K}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{op}}\otimes_k A).$$

These imply that $\mathsf{K}(\mathsf{mod}\, A) \overset{t}{\cong} \mathsf{K}(\mathsf{mod}\, B)$. Let $M = \Omega^n(\mathrm{Hom}_A(S^{-n}, A))$, the $n$th syzygy as as a $B^{\mathrm{op}}\otimes_k A$-module and $N = \Omega^{-n}(S^{-n})$, the $-n$th syzygy as as a $A^{\mathrm{op}}\otimes_k B$-module. Since $\mathrm{Hom}_A(S^{-n}, A)$ is $A$-projective and $B$-projective, $M$ is $A$-projective and $B$-projective. Similarly, $N$ is $A$-projective and $B$-projective. Then

$$-\otimes_A M : \mathsf{mod}\, A \to \mathsf{mod}\, B \quad \text{and} - \otimes_B N : \mathsf{mod}\, B \to \mathsf{mod}\, A$$

induce triangle functors between $\underline{\mathsf{mod}}A$ and $\underline{\mathsf{mod}}B$. By Lemma 17.19, all terms but the term $\mathrm{Hom}_A(S^{-n}, A)\otimes_A S^{-n}$ of a double complex $\mathrm{Hom}_A(S^{\bullet}, A) \ddot{\otimes}_B S^{\bullet}$ are $A^{\mathrm{op}}\otimes_k A$-projective. Therefore $A$ is a direct summand of $\bigoplus_{p=q} \mathrm{Hom}_A(S^p, A)\otimes_B S^q$ as a $A^{\mathrm{op}}\otimes_k A$-module. For each $X \in \mathsf{mod}\, A$, we have isomorphisms in $\underline{\mathsf{mod}}A$

$$\begin{aligned} X &\cong X\otimes_A A \\ &\cong X\otimes_A \mathrm{Hom}_A(S^{-n}, A)\otimes_B S^{-n} \\ &\cong \omega^{-n}(X\otimes_A M\otimes_B S^{-n}) \\ &\cong \omega^n\omega^{-n}(X\otimes_A M\otimes_B N) \\ &\cong X\otimes_A M\otimes_B N, \end{aligned}$$

where $\omega$ is the loop space functor on $\underline{\mathsf{mod}}A$. Similarly, for each $Y \in \mathsf{mod}\,B$, we have an isomorphism in $\underline{\mathsf{mod}}B$

$$Y \cong Y \otimes_B N \otimes_A M.$$

$\square$

**Proposition 17.21.** *Let $A$ and $B$ be indecomposable symmetric $k$-algebras, $X^{\bullet}$ a biperfect complex in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,B^{\mathrm{op}} \otimes_k A)$, and $X^{\vee\bullet} = \mathrm{Hom}^{\bullet}_k(X^{\bullet}, k)$. If $X^{\bullet} \dot{\otimes}^{\boldsymbol{L}}_A X^{\vee\bullet} \cong B$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,B^{\mathrm{op}} \otimes_k B)$, then $X^{\bullet}$ is a two-sided tilting complex.*

*Proof.* Since we have isomorphisms

$$\begin{aligned} X^{\vee\bullet} &\cong \boldsymbol{R}\mathrm{Hom}^{\bullet}_A(X^{\bullet}, A) \\ &\cong \boldsymbol{R}\mathrm{Hom}^{\bullet}_B(X^{\bullet}, B), \end{aligned}$$

by Proposition 15.5, lemma 17.12, $F = - \dot{\otimes}^{\boldsymbol{L}}_B X^{\bullet} : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,B) \to \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)$ and $F^{\vee} = - \dot{\otimes}^{\boldsymbol{L}}_A X^{\vee\bullet} : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A) \to \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,B)$ are both left and right adjoint to one another. By adjunction arrows of adjoint pairs $(F, F^{\vee})$, $(F^{\vee}, F)$, we have morphisms $\alpha : A \to X^{\vee\bullet} \dot{\otimes}^{\boldsymbol{L}}_B X^{\bullet}$, $\beta : X^{\vee\bullet} \dot{\otimes}^{\boldsymbol{L}}_B X^{\bullet} \to A$. By Proposition 1.17, we have a split monomorphism $\alpha F : X^{\bullet} \to X^{\bullet} \dot{\otimes}^{\boldsymbol{L}}_A X^{\vee\bullet} \dot{\otimes}^{\boldsymbol{L}}_B X^{\bullet}$, a split epimorphism $\beta F : X^{\bullet} \dot{\otimes}^{\boldsymbol{L}}_A X^{\vee\bullet} \dot{\otimes}^{\boldsymbol{L}}_B X^{\bullet} \to X^{\bullet}$. Since $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,B^{\mathrm{op}} \otimes_k A)$ is a Krull-Schmidt category by Corollary 11.19, $X^{\bullet} \dot{\otimes}^{\boldsymbol{L}}_A X^{\vee\bullet} \cong B$ implies that $\alpha F$, $\beta F$ are isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,B^{\mathrm{op}} \otimes_k A)$. If $\beta\alpha$ is not an isomorphism, then this contradicts $(\beta\alpha)F$ is an isomorphism. Therefore $A$ is a direct summand of $X^{\vee\bullet} \dot{\otimes}^{\boldsymbol{L}}_B X^{\bullet}$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A^{\mathrm{op}} \otimes_k A)$. Since $X^{\vee\bullet} \dot{\otimes}^{\boldsymbol{L}}_B X^{\bullet} \dot{\otimes}^{\boldsymbol{L}}_A X^{\vee\bullet} \dot{\otimes}^{\boldsymbol{L}}_B X^{\bullet} \cong X^{\vee\bullet} \dot{\otimes}^{\boldsymbol{L}}_B X^{\bullet}$, we have $A \cong X^{\vee\bullet} \dot{\otimes}^{\boldsymbol{L}}_B X^{\bullet}$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A^{\mathrm{op}} \otimes_k A)$. $\square$

**Proposition 17.22.** *Let $A$ be a symmetric $k$-algebra which has no simple projective $A$-module, $e$ an idempotent of $A$, and $P^{\bullet} : P^{-1} \xrightarrow{d^{-1}} P^0 = Ae \otimes_k eA \xrightarrow{\mu} A$, where $\mu$ is the multiplication morphism. Then $P^{\bullet}$ is a tilting complex. Moreover, ${}_AP^{\bullet}_A$ is a two-sided tilting complex if and only if $\dim_k eAe = 2$.*

*Proof.* Since $D_k P^{\bullet} = (D_k A \to D_k(Ae \otimes_k eA)) \cong (A \to Ae \otimes_k eA)$, $\mathrm{Hom}^{\bullet\bullet}_A(P^{\bullet}_A, P^{\bullet}_A) \cong P^{\bullet} \ddot{\otimes}_A D_k P^{\bullet}$ has the form

$$\begin{array}{ccc} Ae \otimes_k eA & \longrightarrow & A \\ \downarrow & & \downarrow \\ Ae \otimes_k eAe \otimes_k eA & \longrightarrow & Ae \otimes_k eA, \end{array}$$

where the left vertical arrow is monic and the bottom horizontal arrow is epic. Then $\mathrm{H}^i(\mathrm{Hom}^{\bullet}_A(P^{\bullet}_A, P^{\bullet}_A)) = 0$ for $i \neq 0$. Since

$$P^{\bullet} = (eAe \otimes_k eA \to eA) \oplus ((1-e)Ae \otimes_k eA \to (1-e)A)$$

and $\dim_K eAe = n \geq 2$,

$$P^{\bullet} \cong (eA^{n-1} \to O) \oplus ((1-e)Ae \otimes_k eA \to (1-e)A),$$

and then $P^\cdot$ generates $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A_A)$. By the above diagram, we have an isomorphism in $\mathsf{mod}\, A^{\mathrm{op}}\otimes_k A$

$$Ae\otimes_k eA\oplus \mathrm{H}^0(P^\cdot \overset{\cdot}{\otimes}_A D_k P^\cdot)\oplus Ae\otimes_k eA \cong Ae\otimes_k eAe\otimes_k eA\oplus A.$$

By the Krull-Schmidt Theorem, we have

$$\mathrm{H}^0(P^\cdot \overset{\cdot}{\otimes}_A D_k P^\cdot) \cong A\oplus(Ae\otimes_k eA)^{n-2}.$$

□

## 18. Cotilting Bimodule Complexes

Throughout this section, unless otherwise stated, $k$ is a commutative ring, $A, B$, $C$ are $k$-algebras which are $k$-projective modules. See Propositions 15.7, 15.8.

**Definition 18.1** (Cotilting Bimodule Complexes)**.** Let A be a right coherent $k$-algebra and B a left coherent $k$-algebra. A complex ${}_BU^\cdot_A \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B^{\mathrm{op}}\otimes_k A)$ is called a *cotilting B-A-bimodule complex* provided that it satisfies

1. $Res_A U^\cdot \in \mathsf{D}^{\mathrm{b}}_c(\mathsf{Mod}\, A)_{\mathrm{fid}}$ and $Res_{B^{\mathrm{op}}} U^\cdot \in \mathsf{D}^{\mathrm{b}}_c(\mathsf{Mod}\, B^{\mathrm{op}})_{\mathrm{fid}}$.
2. $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(U^\cdot, U^\cdot[i]) = 0$ for all $i \neq 0$.
3. $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, B^{\mathrm{op}})}(U^\cdot, U^\cdot[i]) = 0$ for all $i \neq 0$.
4. the left multiplication morphism $B \to \mathrm{End}_{\mathsf{D}(\mathsf{Mod}\, A)}(U^\cdot)$ is a ring isomorphism.
5. the right multiplication morphism $A \to \mathrm{End}_{\mathsf{D}(\mathsf{Mod}\, B^{\mathrm{op}})}(U^\cdot)^{\mathrm{op}}$ is a ring isomorphism.

In case of $B = A$, we will call a cotilting $A$-$A$-bimodule complex a *dualizing A-bimodule complex.*

**Proposition 18.2.** *Let $A$ be a right coherent $k$-algebra and $B$ a left coherent $k$-algebra, and ${}_BU^\cdot_A \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B^{\mathrm{op}}\otimes_k A)$ a cotilting B-A-bimodule complex. Then*

$$\boldsymbol{R}^* \mathrm{Hom}^\cdot_A(-, U^\cdot) : \mathsf{D}^*_c(\mathsf{Mod}\, A) \to \mathsf{D}^\dagger_c(\mathsf{Mod}\, B^{\mathrm{op}})$$

*and*

$$\boldsymbol{R}^\dagger \mathrm{Hom}^\cdot_B(-, U^\cdot) : \mathsf{D}^\dagger_c(\mathsf{Mod}\, B^{\mathrm{op}}) \to \mathsf{D}^*_c(\mathsf{Mod}\, A)$$

*induce the duality, where $(*, \dagger) =$ (nothing, nothing), $(+,-), (-,+), (b, b)$.*

*Proof.* Since $Res_A U^\cdot \in \mathsf{D}^{\mathrm{b}}_c(\mathsf{Mod}\, A)_{\mathrm{fid}}$, by Proposition 10.21 $\boldsymbol{R}^* \mathrm{Hom}^\cdot_A(-, U^\cdot)$ is way-out in both directions. Since $\boldsymbol{R}^* \mathrm{Hom}^\cdot_A(A, U^\cdot) \cong Res_{B^{\mathrm{op}}} U^\cdot \in \mathsf{D}^\dagger_c(\mathsf{Mod}\, B^{\mathrm{op}})$,we have $\boldsymbol{R}^* \mathrm{Hom}^\cdot_A(-, U^\cdot) : \mathsf{D}^*_c(\mathsf{Mod}\, A) \to \mathsf{D}^\dagger_c(\mathsf{Mod}\, B^{\mathrm{op}})$ by Proposition 12.12. Similarly we have $\boldsymbol{R}^\dagger \mathrm{Hom}^\cdot_B(-, U^\cdot) : \mathsf{D}^\dagger_c(\mathsf{Mod}\, B^{\mathrm{op}}) \to \mathsf{D}^*_c(\mathsf{Mod}\, A)$. Since

$$(\boldsymbol{R}^* \mathrm{Hom}^\cdot_A(-, U^\cdot), \boldsymbol{R}^\dagger \mathrm{Hom}^\cdot_B(-, U^\cdot))$$

is a right adjoint pair, we have adjunction arrows

$$\eta : \mathbf{1}_{\mathsf{D}^*_c(\mathsf{Mod}\, A)} \to \boldsymbol{R}^\dagger \mathrm{Hom}^\cdot_B(-, U^\cdot) \circ \boldsymbol{R}^* \mathrm{Hom}^\cdot_A(-, U^\cdot)$$
$$\theta : \mathbf{1}_{\mathsf{D}^*_c(\mathsf{Mod}\, B^{\mathrm{op}})} \to \boldsymbol{R}^* \mathrm{Hom}^\cdot_A(-, U^\cdot) \circ \boldsymbol{R}^\dagger \mathrm{Hom}^\cdot_B(-, U^\cdot).$$

It is not hard that we have a commutative diagram in $\mathsf{D}^*_c(\mathsf{Mod}\, A^{\mathrm{op}}\otimes_K A)$

$$\begin{array}{ccc} A & \xrightarrow{\eta(A)} & \boldsymbol{R}^\dagger \mathrm{Hom}^\cdot_B(\boldsymbol{R}^* \mathrm{Hom}^\cdot_A(A, U^\cdot), U^\cdot) \\ \Vert & & \downarrow\wr \\ A & \xrightarrow{\rho_A} & \boldsymbol{R}^\dagger \mathrm{Hom}^\cdot_B(U^\cdot, U^\cdot), \end{array}$$

Then $\eta(A)$ is an isomorphism. By Proposition 12.11, $\eta$ is an isomorphism. Similarly, $\theta$ is an isomorphism. □

**Lemma 18.3** (Piled Resolutions Lemma)**.** *Let $\mathcal{A}$ be an abelian category satisfying the condition Ab4* with enough injectives, and let $C^{\cdot\cdot}$ be a double complex with $C^{p,q} = 0$ ($p < 0$ or $q < 0$), $C^{\cdot j} \to I^{\cdot j}$ quasi-isomorphisms with $I^{\cdot j} : 0 \to I^{-s,j} \to I^{-s+1,j} \to \dots \in \mathsf{K}^+(\mathsf{Inj}\,\mathcal{A})$ for all $i$. Then, there is a quasi-isomorphism from $\mathrm{Tot}\, C^{\cdot\cdot} = \overset{\wedge}{\mathrm{Tot}} C^{\cdot\cdot}$ to a complex $J^{\cdot}$ of the following form in $\mathsf{K}^+(\mathsf{Inj}\,\mathcal{A})$*

$$J^n = \begin{cases} O \text{ if } n < -s \\ \bigoplus_{i+j=n} I^{i,j} \text{ if } n \geq -s \end{cases}$$

*Proof.* For a double complex $C^{\cdot\cdot}$, we have a sequence of morphisms $\{\mathrm{Tot}\,\tau^{\mathrm{II}}_{\leq n} C^{\cdot\cdot} \to \mathrm{Tot}\,\tau^{\mathrm{II}}_{\leq n-1} C^{\cdot\cdot}\}$. By induction on $n \geq 0$, we construct complexes $V^{\cdot}_n$ and morphisms of triangles in $\mathsf{K}^+(\mathcal{A})$

$$\begin{array}{ccccccc} \mathrm{Tot}\,\tau^{\mathrm{II}}_{\leq n} C^{\cdot\cdot} & \longrightarrow & \mathrm{Tot}\,\tau^{\mathrm{II}}_{\leq n-1} C^{\cdot\cdot} & \longrightarrow & C^{\cdot n}[1-n] & \longrightarrow & \mathrm{Tot}\,\tau^{\mathrm{II}}_{\leq n} C^{\cdot\cdot}[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ V^{\cdot}_n & \longrightarrow & V^{\cdot}_{n-1} & \longrightarrow & I^{\cdot j}[1-n] & \longrightarrow & V^{\cdot}_n[1] \end{array}$$

where all vertical arrows are quasi-isomorphisms. If $n = 0$, then we take $V^{\cdot}_0 = I^{\cdot 0}$. Let $n > 0$. Since $V^{\cdot}_{n-1} \to \mathrm{Tot}\,\tau^{\mathrm{II}}_{\leq n-1} C^{\cdot\cdot}$ and $C^{\cdot n}[1-n] \to I^{\cdot n}[1-n]$ are quasi-isomorphisms, we choose a morphism $V^{\cdot}_{n-1} \xrightarrow{g_{n-1}} I^{\cdot j}[1-n]$ in $\mathsf{C}(\mathcal{A})$ such that

$$\begin{array}{ccc} \mathrm{Tot}\,\tau^{\mathrm{II}}_{\leq n-1} C^{\cdot\cdot} & \longrightarrow & C^{\cdot n}[1-n] \\ \downarrow & & \downarrow \\ V^{\cdot}_{n-1} & \xrightarrow{g_{n-1}} & I^{\cdot j}[1-n] \end{array}$$

is commutative in $\mathsf{K}^+(\mathcal{A})$. We take $V^{\cdot}_n = \mathrm{M}^{\cdot}(g_{n-1})[-1]$ in $\mathsf{C}(\mathcal{A})$. Then $V^{\cdot}_n \to V^{\cdot}_{n-1}$ is a term-split epimorphism and have the above morphism of triangles. By Proposition 11.7, we have isomorphisms in $\mathsf{D}^+(\mathcal{A})$

$$\begin{aligned} \mathrm{Tot}\, C^{\cdot\cdot} &= \overset{\wedge}{\mathrm{Tot}} C^{\cdot\cdot} \\ &\cong \varprojlim \mathrm{Tot}\,\tau^{\mathrm{II}}_{\leq n} C^{\cdot\cdot} \\ &\cong \underset{\longleftarrow}{\mathrm{hlim}}\ \mathrm{Tot}\,\tau^{\mathrm{II}}_{\leq n} C^{\cdot\cdot} \\ &\cong \underset{\longleftarrow}{\mathrm{hlim}}\ V^{\cdot}_n \\ &\cong \varprojlim V^{\cdot}_n. \end{aligned}$$

By the construction of $V^{\cdot}_n$, we have the required complex. □

**Theorem 18.4.** *Let $A$ be a right noetherian $k$-algebra and $B$ a left coherent $k$-algebra, and ${}_B U^{\cdot}_A \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B^{\mathrm{op}} \otimes_k A)$ a cotilting $B$-$A$-bimodule complex. If $I^{\cdot} \in \mathsf{K}^+(\mathsf{Inj}\, A)$ is quasi-isomorphic to $Res_A U^{\cdot}$ in $\mathsf{K}^+(\mathsf{Mod}\, A)$, then $I^{\cdot}$ contains all indecomposable injective $A$-modules.*

*Proof.* According to Proposition 18.2 and Example 10.14, for any $X^{\cdot} \in \mathsf{D}^{\mathrm{b}}_c(\mathsf{Mod}\, A)$, there exists $P^{\cdot} \in \mathsf{K}^-(\mathsf{proj}\, B^{\mathrm{op}})$ such that

$$\begin{aligned} \mathrm{Hom}^{\cdot}_B(P^{\cdot}, U^{\cdot}) &\cong \boldsymbol{R}\mathrm{Hom}^{\cdot}_B(P^{\cdot}, U^{\cdot}) \\ &\cong X^{\cdot}. \end{aligned}$$

For any $P^i$, it is easy to see $\mathrm{Hom}_B(P^i, U^\cdot) \in \mathsf{add}\, U^\cdot_A$. Then $\mathrm{Hom}_B(P^i, U^\cdot) \in \mathsf{add}\, I^\cdot$. By piled resolutions lemma, we have a quasi-isomorphism $X^\cdot \to J^\cdot$ in $\mathsf{K}^+(\mathsf{Mod}\, A)$ with all $J^i \in \mathsf{Add}\left(\coprod_{i\in\mathbb{Z}} I^i\right)$. Since $A$ is noetherian, $J^i$ is $A$-injective. We take $X^\cdot = A/\mathfrak{p}$ where $\mathfrak{p}$ is any right ideal of $A$. Since $J^\cdot$ in $\mathsf{K}^+(\mathsf{Mod}\, A)$, it is easy to see that we have a monomorphism $A/\mathfrak{p} \to J^0$. Hence the injective envelope $\mathrm{E}(A/\mathfrak{p})$ is a direct summand of $J^0$. □

**Corollary 18.5** (Like-Corollary)**.** *Let $A$ be a right noetherian and left coherent ring with* inj dim ${}_AA$, inj dim $A_A < \infty$. *Then any injective resolution of a right $A$-module $A_A$ contains all indecomposable injective $A$-modules.*

*Proof.* By the same technique in Proposition 18.2, $\boldsymbol{R}^{\mathrm{b}} \mathrm{Hom}_A(-, A) : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A) \to \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A^{\mathrm{op}})$ and $\boldsymbol{R}^{\mathrm{b}} \mathrm{Hom}_A(-, A) : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A^{\mathrm{op}}) \to \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$ induce a duality. By the above proof, we get the statement. □

## References


[BBD] A. A. Beilinson, J. Bernstein and P. Deligne, Faisceaux Pervers, Astérisque **100** (1982).

[BN] M. Böckstedt and A. Neeman, Homotopy Limits in Triangulated Categories, Compositio Math. **86** (1993), 209-234.

[CE] H. Cartan, S. Eilenberg, “Homological Algebra,” Princeton Univ. Press, 1956.

[CW] S. Mac Lane, “Categories for the Working Mathematician,” GTM **5**, Springer-Verlag, Berlin, 1972.

[Ha] D. Happel, “Triangulated Categories in the Representation Theory of Finite-Dimensional Algebras,” London Math. Soc. Lecture Notes **119**, University Press, Cambridge, 1987.

[HS] P. J. Hilton, U. Stammbach, “A Course in Homological Algebra,” GTM **4**, Springer-Verlag, Berlin, 1971.

[Ho] M. Hoshino, Derived Categories, Seminar Note, 1997.

[HK] M. Hoshino and Y. Kato, Tilting Complexes Defined by Idempotents, preprint.

[Ke1] B. Keller, Deriving DG Categories, Ann. Scient. Ec. Norm. Sup. **27** (1994), 63 - 102.

[Ke2] B. Keller, Invariance and Localization for Cyclic Homology of DG algebras, Journal of Pure and Applied Algebra, **123** (1998), 223-273.

[KV] B. Keller, D. Vossieck, Sous Les Catégorie Dérivées, C. R. Acad. Sci. Paris **305** (1987), 225-228.

[ML] S. Mac Lane, “Homology,” Springer-Verlag, Berlin, 1963.

[Mi1] J. Miyachi, Localization of Triangulated Categories and Derived Categories, J. Algebra **141** (1991), 463-483.

[Mi2] J. Miyachi, Duality for Derived Categories and Cotilting Bimodules, J. Algebra **185** (1996), 583 - 603.

[Mi3] J. Miyachi, Derived Categories and Morita Duality Theory, J. Pure and Applied Algebra **128** (1998), 153-170.

[Mi4] J. Miyachi, Injective Resolutions of Noetherian Rings and Cogenerators, Proceedings of The AMS **128** (2000), no. 8, 2233-2242.

[Po] N. Popescu, “Abelian Categories with Applications to Rings and Modules, ” Academic Press, London-New York, 1973.

[Qu] D. Quillen, “Higher Algebraic K-theory I,” pp. 85-147, Lecture Notes in Math. **341**, Springer-Verlag, Berlin, 1971.

[RD] R. Hartshorne, “Residues and Duality,” Lecture Notes in Math. **20**, Springer-Verlag, Berlin, 1966.

[Rd1] J. Rickard, Morita Theory for Derived Categories, J. London Math. Soc. **39** (1989), 436-456.

[Rd2] J. Rickard, Derived Equivalences as Derived Functors, J. London Math. Soc. **43** (1991), 37-48.

[Rd3] J. Rickard, Splendid Equivalences: Derived Categories and Permutation Modules, Proc. London Math. Soc. **72** (1996), 331-358.

[Rl] C.M. Ringel, “Tame Algebras and Integral Quadratic Forms,” Lecture Notes in Math. **1099**, Springer-Verlag, Berlin, 1984.

[RZ] R. Rouquier and A. Zimmermann, Picard Groups for Derived Module Categories, preprint.

[We] C. A. Weibel, “An Introduction to Homological Algebra,” Cambridge studies in advanced mathematics. **38**, Cambridge Univ. Press, 1995.

[Sp] N. Spaltenstein, Resolutions of Unbounded Complexes, Composition Math. **65** (1988), 121-154.

[Ye] A. Yekutieli, Dualizing Complexes over Noncommutative Graded Algebras, J. Algebra **153** (1992), 41-84.

[Ve] J. Verdier, “Catéories Déivées, état 0”, pp. 262-311, Lecture Notes in Math. **569**, Springer-Verlag, Berlin, 1977.

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