Modules of the Highest Homological Dimension over a Gorenstein Ring

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Dedicated to Professor Kent R. Fuller on his 60th birthday

We will study modules of the highest injective, projective and flat dimension over a Gorenstein ring. Let \( R \) be a Gorenstein ring of self-injective dimension \( n \) and \( 0 \to R \to E_0 \to \cdots \to E_n \to 0 \) a minimal injective resolution. Then it is shown in [F-I] that the flat dimension and projective dimension of \( E_n \) is \( n \), the highest dimension. In this note, we shall prove that if \( M \) is a left \( R \)-module of injective dimension \( n \), then the last injective term \( E^n(M) \) in a minimal injective resolution of \( M \) has projective and flat dimension \( n \), and any indecomposable summand of \( E^n(M) \) embeds in \( E_n \). As a consequence, we obtain that if \( R \) is Auslander-Gorenstein, then \( E^n(M) \) has essential socle.

1. Introduction

A Noether ring \( R \) is called Gorenstein if \( R \) has left and right finite self-injective dimensions. Further, a Noether ring \( R \) is called Auslander-Gorenstein if \( R \) is Gorenstein and in a minimal injective resolution \( 0 \to R \to E_0 \to E_1 \to \cdots \), each \( E_i \) has flat dimension at most \( i \). This concept was introduced by Auslander as

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a non-commutative version of the Gorenstein condition for commutative rings, studied by Bass [Ba]. In the non-commutative case, we can also see the ubiquity of Auslander-Gorenstein rings in several articles, for example, [B-G], [G-L], [Le], [L-S], [S-Z].

We showed in [Iw1] that for a Gorenstein ring of self-injective dimension $n$, finiteness of the injective, projective and flat dimensions of a module are all equivalent, and all of these dimensions are at most $n$. This fact motivated our interest in modules with the highest injective, projective or flat dimension. Let $R$ be a Gorenstein ring of self-injective dimension $n$ and $0 \rightarrow R \rightarrow E_0 \rightarrow \cdots \rightarrow E_n \rightarrow 0$ a minimal injective resolution. Then it is shown in [F-I] that any direct summand of $E_n$ has the highest projective and flat dimension $n$. In this note, we will study the relationship between more general modules of projective (or flat) dimension $n$ and the module $E_n$.

Throughout this note, $\text{id}(M)$, $\text{pd}(M)$ and $\text{fd}(M)$ stand for the injective, projective and flat dimension of a module $M$, respectively. Further, $0 \rightarrow M \rightarrow E^0(M) \rightarrow \cdots \rightarrow E^n(M) \rightarrow \cdots$ is a minimal injective resolution of $M$.

The results obtained in this note are the following.

**Theorem 1.** Let $R$ be a Gorenstein ring of self-injective dimension $n$ and $0 \rightarrow R \rightarrow E_0 \rightarrow \cdots \rightarrow E_n \rightarrow 0$ a minimal injective resolution. If a left $R$-module $M$ has injective dimension $n$, then any indecomposable direct summand $E$ of $E^n(M)$ is isomorphic to $E$ and has projective and flat dimension $n$.

As a byproduct, [Mi2, Corollary 1.3] and [I-S2, Theorem 6] yield a generalization of [I-S2, Theorem 6] for Auslander-Gorenstein rings. Hoshino showed that an injective indecomposable module of flat dimension $i$ over an Auslander-Gorenstein ring appears in $i$-th injective term of a minimal injective resolution of the ring ([Ho, Theorem 6.3]). Miyachi showed that any injective indecomposable module over a Gorenstein ring appears in some injective term of a minimal injective resolution of the ring ([Mi1, Corollary 4.7]).

**Theorem 2.** If $R$ is an Auslander-Gorenstein ring of self-injective dimension $n$, then any injective indecomposable left $R$-module of flat dimension $n$ is isomorphic to a direct summand of $E_n$ and is
of the form $E(S)$ for a simple left module $S$. Thus if a left $R$-module $M$ has injective dimension $n$, $E^n(M)$ has essential socle.

The final result generalizes [I-S1, Theorem; I-S2, Theorem 2]. It appears interesting to study the distribution of injective indecomposables along the terms of a minimal injective resolution of a Gorenstein ring.

**Proposition 3.** Let $R$ be a Noether ring and $0 \to R \to E_0 \to \cdots \to E_n \to \cdots$ a minimal injective resolution of $R$.

1. If $M$ is a left $R$-module with $0 < i = \text{id}(M) < \infty$, then $E_0$ and $E_i(M)$ have no isomorphic direct summands in common.

2. If $R$ has left self-injective dimension $n \geq 1$, then $E_0$ and $E_n$ have no isomorphic direct summands in common.

### 2. The Proofs

**Proof of Theorem 1.**

By [Iw1, Theorem 2], $M$ has projective dimension at most $n$. Thus let $0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ be a projective resolution of $M$ and consider an injective resolution of each $P_i$ ($0 \leq i \leq n$)

$$0 \to P_i \to E^0(P_i) \to E^1(P_i) \to \cdots \to E^n(P_i) \to 0.$$

Then $E^j(P_i)$ for each $j$ ($0 \leq j \leq n$) is a direct summand of a direct sum of copies of $E_j$. Hence by [Mi2, Corollary 1.3], $M$ has an injective resolution of the following form

$$0 \to M \to Q \to \bigoplus_{i=0}^{n-1} E^{i+1}(P_i) \to \bigoplus_{i=0}^{n-2} E^{i+2}(P_i) \to \cdots \to E^{n-1}(P_0) \oplus E^n(P_1) \to E^n(P_0) \to 0.$$ 

Here $Q$ is a direct summand of $\bigoplus_{i=0}^n E^i(P_i)$. Then $E^n(M)$ is a direct summand of $E^n(P_0)$, and so a direct summand of direct sum of copies of $E_n$. Since any indecomposable summand $E$ of $E^n(M)$ is uniform, $E$ embeds in $E_n$. □
Proof of Theorem 2.
Let $E$ be an injective indecomposable left module of flat dimension $n$. By [Mi1, Corollary 4.7], $E$ is isomorphic to a direct summand in $E_n$. Since $\text{Soc}(E_n)$ is essential in $E_n$ ([I-S 2, Theorem 6]), $E$ is of the form $E(S)$ for some simple module $S$.

By Theorem 1 and [F-I, Proposition 1.1], any direct summand of $E^n(M)$ has flat dimension $n$ and so has essential socle. That is, the socle of $E^n(M)$ is essential. \qed

Proof of Proposition 3.
(1) Let $U$ be any nonzero submodule of $E_0$ and $V = U \cap R \neq 0$. Then from the exact sequence

$$0 \rightarrow V \rightarrow R \rightarrow R/V \rightarrow 0,$$

we have an exact sequence

$$\text{Ext}^i_R(R, M) \rightarrow \text{Ext}^i_R(V, M) \rightarrow \text{Ext}^{i+1}_R(R/V, M).$$

Here $\text{Ext}^i_R(R, M) = 0$ from $i > 0$ and $\text{Ext}^{i+1}_R(R/V, M) = 0$ from $\text{id}(M) = i$. Hence we obtain $\text{Ext}^i_R(V, M) = 0$ and thus we see that $V$ is not monomorphic to $E^n(M)$.

(2) is obvious from (1). \qed

3. Examples

Let us conclude this note with a few examples. In the following examples, if $R$ is a path algebra given by a quiver $Q$ with set $Q_0$ of vertices and $i \in Q_0$, then $S(i)$ denotes the simple $R$-module corresponding to the vertex $i$ and $E(i)$ its injective hull.

(1) Theorem 1 and Proposition 3 prompt us to raise the following question: Let $R$ be a Gorenstein ring of self-injective dimension $n$ and $E$ an injective indecomposable $R$-module of projective dimension $n$. Then does there exist an $R$-module $M$ of injective dimension $n$ such that $E$ embeds in $E^n(M)$? It’s easy to see that the question is affirmative if $R$ is Auslander-Gorenstein. However, the answer is negative for Gorenstein rings. For example, let $R$ be
a finite dimensional algebra over any field given by the following quiver

\[ \begin{array}{c}
1 \\
\alpha \\
\gamma \\
3 \xrightarrow{\gamma} 4 \xrightarrow{\delta} 5 \\
\beta \\
2
\end{array} \]

with the relations $\gamma \alpha = \gamma \beta = \epsilon \delta = \delta \epsilon = 0$. Then $R$ is a Gorenstein ring of self-injective dimension 2 and has infinite global dimension. $E(3)$ has projective dimension 2 but never appears in $E^2(M)$ for any $R$-module $M$ of injective dimension 2. Also we can see from this observation that an injective indecomposable module with the highest projective dimension does not necessarily embed in the last term of a minimal injective resolution of a Gorenstein ring.

Moreover, $E(1)$ and $E(2)$ are both direct summands of the last injective term $E_2$ in a minimal injective resolution $0 \to R \to E_0 \to E_1 \to E_2 \to 0$ but $\Ext^1_R(E(i), R) \neq 0$ ($i = 1, 2$). Hence $E(1)$ and $E(2)$ are not holonomic. Here a finitely generated module $X$ over a Gorenstein ring $R$ of self-injective dimension $n$ is called holonomic if $\Ext^n_R(X, R) = 0$ for all $i \neq n$.

Finally we can see in this example that all injective terms $E_0$, $E_1$ and $E_2$ have the highest projective and flat dimensions.

(2) In [Iw2], it is proved that any holonomic module over an Auslander-Gorenstein ring has finite composition length and embeds in a direct sum of finitely many copies of the last injective term in a minimal injective resolution of the ring. However a submodule of finite composition length in the last injective term is not necessarily holonomic.

For example, let $R$ be a finite dimensional algebra over any field given by the following quiver

\[ \begin{array}{c}
2 \\
\alpha \\
1 \xrightarrow{\mu} 5 \xrightarrow{\lambda} 4 \\
\beta \\
3 \xrightarrow{\delta}
\end{array} \]

with the relations $\mu \lambda = \alpha \mu = \beta \mu = 0$ and $\gamma \alpha = \delta \beta$. Then $R$ is Auslander-Gorenstein of self-injective dimension 4.
Consider the left $R$-module $M$ of dimension vector $(0, 1, 1, 1, 0)$, then $M$ is a submodule of the last injective term of a minimal injective resolution of $R \otimes R$ but not holonomic. For, we can see $\text{Ext}^1_R(M, R) \neq 0$, that is, the grade of $M$ is one.

(3) We can see that if $R$ is a Gorenstein ring of self-injective dimension $n$ and $S$ is a simple submodule of the last injective term $E_n$ in a minimal injective resolution of $R \otimes R$, then $\text{pd}(S) = \text{fd}(S) = n$ or $\infty$. Conversely, if $S$ is a simple $R$-module of the highest projective dimension $n$, $S$ appears in the socle of $E_n$. There is an example of a Gorenstein ring $R$ with a simple module of infinite projective and flat dimension not appearing in $E_n$.

Let $R$ be a finite dimensional algebra over any field given by the following quiver

\[
\begin{array}{ccc}
1 & \xymatrix{\delta} & 3 \\
\alpha & 2 & \gamma \\
\beta & & \\
\end{array}
\]

with the relations $\alpha \delta = \gamma \alpha = \beta^2 = 0$. Then $R$ is Auslander-Gorenstein of self-injective dimension 3. We can see

$$\text{pd}(S(1)) = 2, \quad \text{pd}(S(2)) = \infty, \quad \text{pd}(S(3)) = 3$$

and

$$E_0 = E(1)^{(4)}, \quad E_1 = E(2)^{(2)}, \quad E_2 = E(1), \quad E_3 = E(3).$$

Here, $M^{(t)}$ stands for a direct sum of $t$ copies of a module $M$.

References


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