Injective Resolutions of Complexes, Modules over Noetherian Rings

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\(k\) : a commutative ring, \(R\) : a ring with nonzero unity
\(A, B, R\) : projective \(k\)-algebras (i.e. \(k\)-algebras which are finitely generated \(k\)-modules)

**\(R\)-Mod** (resp., \(R\)-mod, \(R\)-Inj) : the category of left (resp., finitely presented left, injective left) \(R\)-modules.

For a left \(R\)-module \(M\),

\(\text{idim}_R M, \text{fdim}_R M, \text{pdim}_R M\) : the injective, flat, projective dimensions of \(M\)

\(E(M)\) : the injective hull of \(M\)

\(0 \to R \to E^0 \to E^1 \to E^2 \to \cdots\) : the minimal injective resolution of \(R\)

\(0 \to M \to E^0(M) \to E^1(M) \to E^2(M) \to \cdots\) : the minimal injective resolution of \(M\)

\(\oplus := \) is isomorphic to a direct summand of

**\(\text{Add} M\)** (resp., **\(\text{add} M\)**) : the category of left \(R\)-modules which are direct summands of direct sums of copies of \(M\) (resp., direct summands of finite direct sums of copies of \(M\))

\(\mathcal{A}\) : an abelian category, \(\mathcal{B}\) : an additive subcategory of \(\mathcal{A}\)

**\(K(\mathcal{B})\)** : the homotopy category of \(\mathcal{B}\).

**\(K^-(\mathcal{B}), K^+(\mathcal{B})\)** and **\(K^b(\mathcal{B})\)** : the full subcategories of **\(K(\mathcal{B})\)** generated by bounded below complexes, bounded above complexes, **\(K(\mathcal{B})\)**, respectively.

**\(D^b(\mathcal{A})\)** : the derived category of **\(\mathcal{A}\)**

\(\tau^\infty X^* : \cdots \to X^{n+1} \to X^n \to 0 \to \cdots\)

### 1. INJECTIVE RESOLUTIONS OF COMPLEXES

The piled resolution Lemma [Mi2]. Let \(R\) be a ring, and \(L\) a complex \(\ldots \to 0 \to L^0 \to L^1 \to L^2 \to \ldots\) in **\(K(\mathcal{R}-\text{Mod})\)**. If \(0 \to L' \to E'^0 \to E'^1 \to \ldots\) is an injective resolution of \(L'\) (\(i \geq 0\)), then there is a quasi-isomorphism from \(L'\) to a complex of the following form in **\(K(\mathcal{R}-\text{Mod})\)**:

\[\ldots \to 0 \to E'^{0,0} \to \bigoplus_{i+j=1} E'^i \to \bigoplus_{i+j=2} E'^i \to \ldots\]
The piled resolution Lemma'.

Outline of Proof.

We have a sequence of quasi-isomorphisms from \( \tau_{\leq n+1}L^* \) to \( \{ V_k^* \} \). By the construction, it is easy to see that \( \lim_{k \to \infty} V_k^* \) exists. Moreover, we have a quasi-isomorphism \( L^* = \lim_{k \to \infty} \tau_{\leq n}L^* \to \lim_{k \to \infty} V_k^* \).
The switch-back lemma [Mi2]. Let $R$ be a ring, and $0 \to Y_n \to \cdots \to Y_1 \to Y_0 \to X \to 0$ an exact sequence of left $R$-modules. If $0 \to Y_i \to I^{i,0} \to I^{i,1} \to \cdots$ is an injective resolution of $Y_i$ ($0 \leq i \leq n$), then $X$ has the following injective resolution:

$0 \to X \to Q \to \bigoplus_{i+j=1} I^{i,j} \to \bigoplus_{i+j=2} I^{i,j} \to \cdots$,

where $Q$ is a direct summand of $\bigoplus_{i+j=0} I^{i,j}$.

Proof.

2. **Injective Resolutions of Rings and $\Sigma$-Embedding Cogenerators**

For an object $X \in \mathcal{Y}$

$X$: a $\Sigma$-embedding cogenerator (resp., a finitely embedding cogenerator, a cogenerator) for $\mathcal{Y}$

$\Leftrightarrow$ every object in $\mathcal{Y}$ admits an injection to some direct sum (resp., finite direct sum, direct product) of copies of $X$ in $\mathcal{Y}$. 

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Theorem 2.1 ([M2]). Let $R$ be a left Noetherian ring. Then $\oplus_{i=0}^{n} E^i$ is a $\Sigma$-embedding cogenerator for the category of left $R$-modules of projective dimension $\leq n$.

In particular, for an indecomposable injective left $R$-module $I$,
\[ \text{pdim}_R I \leq n \Rightarrow I \langle \oplus_{i=0}^{n} E^i \rangle. \]

Proof. By the switch-back lemma.

Corollary 2.2 ([M2]). Let $R$ be a right coherent and left Noetherian ring. If $\text{idim}_R R$ is finite, then $\oplus_{i=0}^{n} E^i$ is a $\Sigma$-embedding cogenerator for the category of left $R$-modules of finite flat dimension.

Proof. For a left $R$-module $M$, $\text{fdim}_R M < \infty \Rightarrow \text{pdim}_R M < \infty$ [H2, Proposition 6]. Then we complete the proof by Theorem 2.1.

Corollary 2.3 ([M1]). Let $R$ be a right coherent and left Noetherian ring. If $\text{idim}_R R, \text{idm}_R R \leq n$, $\Rightarrow$ every indecomposable injective left $R$-module $I \langle \oplus_{i=0}^{n} E^i \rangle$.

Proof. The flat dimension of every indecomposable injective left $R$-module is at most $n$. By Corollary 2.2, we get the statement.

Corollary 2.4 ([IM], cf. [ASZ]). Let $R$ be an Auslander-Gorenstein ring of self-injective dimension $n$.

For an indecomposable injective left $R$-module $I$, $\text{fdim}_R I = n \Rightarrow I \langle \oplus E^n \rangle$ and $I \cong E(S)$ for a simple left module $S$.

Thus if a left $R$-module $M$ has injective dimension $n$, $E^n(M)$ has essential socle.

Theorem 2.5 ([IM]). Let $R$ be a Gorenstein ring of self-injective dimension $n$, $M$ a left $R$-module.

If $\text{idim}_R M = n$, $\Rightarrow E^n(M) \in \text{Add}(E^n)$. As a consequence, $\text{fdim}_R E = n$ for $E \in \text{Add}(E^n(M))$. 

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3. **Injective Resolutions of Projective Algebras and Cogenerators**

$S_n(A)$ : the **bar resolution** of $A$, i.e. the complex $(S_n(A), d_n : S_{n+1}(A) \rightarrow S_n(A))_{n \geq 0}$ such that $S_n(A)$ is the $(n+2)$-fold tensor product over $k$ of $A$ with itself, and that $d_n(a_0 \otimes \ldots \otimes a_{m-1}) = \sum_{0 \leq i \leq m} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{m-1}$. In this case, $S_n(A)$ is a projective resolution of $A$ in $A \otimes_k A^{\text{op}}$-Mod.

**Theorem 3.1 ([Mi2]).** Let $k$ be a commutative ring, $A$ a $k$-algebra and $B$ a projective $k$-algebra. For an $A$-$B$-bimodule $M$, let $0 \rightarrow \delta M \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \ldots$ be an injective resolution of $M$ as left $A$-modules. Then $M$ has the following injective resolution in $A \otimes_k B^{\text{op}}$-Mod:

$$0 \rightarrow M \rightarrow V^0 \rightarrow V^1 \rightarrow V^2 \rightarrow \ldots,$$

such that $V^n = \bigoplus_{i=0}^n \text{Hom}_k(S_{-i}(B) \otimes_k I^{n-i})$ for all $n \geq 0$, where $S_{-i}(B) = B$.

**Proof.** It is easy to see that we have the following exact sequence in $A \otimes_k B^{\text{op}}$-Mod:
Then \( \chi M_B \cong \text{Hom}_{A M B}(B, S(B), M_B) \) in \( D(A \otimes B^\text{op} \text{-Mod}) \). By adjointness, for every \( n \geq 0 \), we have the following isomorphism in \( A \otimes B^\text{op} \text{-Mod} \):

\[
\text{Hom}_B(S_n(B), \chi M_B) \cong \text{Hom}_{A M B}(B, S_n(B), M_B).
\]

For an injective left module \( I \) over a ring \( R \),

\( \text{Ps}(I) \) : the category of left \( R \)-modules which are direct summands of direct products of copies of \( I \).

**Lemma 3.2** ([Mi2]). Let \( R \) be a ring, \( I \) an injective left \( R \)-module.

(a) If \( R \) is a left Noetherian ring, then \( \text{Ps}(I) \) is closed under direct sums.

(b) If \( R \) is a right coherent ring and \( \text{fdim}_R I \leq n \), then every \( R \)-module of \( \text{Ps}(I) \) is of flat dimension \( \leq n \).

**Proof.** See [Ma] and [CE, Chap. II, Ex. 2].

**Remark.** By Theorem 3.1, \( R \) has the following injective resolution in \( R \otimes R^\text{op} \text{-Mod} \):

\[
0 \to R \to V^0 \to V^1 \to V^2 \to \ldots,
\]

where \( V^n = \bigoplus_{i=0}^n \text{Hom}_k(S_{r-i}(R), E^{\text{op}}) \) for all \( n \geq 0 \). Since all \( S_{r-i}(R) \) are projective \( k \)-modules, it is easy to see that every \( V^n \) belongs to \( \text{Ps}(\bigoplus E^i) \).

**Lemma 3.3** ([Mi2]). Let \( R \) be a left Noetherian ring, \( \{ Q_\lambda, f_{\mu \lambda} : Q_\lambda \to Q_\mu \} \lambda \in \Lambda, \lambda \geq \mu \) a direct system of injective left \( R \)-modules. Then \( \lim_{\lambda \to} Q_\lambda \) belongs to \( \text{Add}(\bigoplus Q_\lambda) \).
Lemma 3.4 ([Mi2]). Let $k$ be a commutative ring, and $R$ a left Noetherian projective $k$-algebra. Then every flat left $R$-module $F$ has the following injective resolution:

$$0 \to F \to I^0 \to I^1 \to I^2 \to \ldots,$$

where $I^n \in \text{Ps}(\bigoplus_{i=0}^n E^i)$ for all $n \geq 0$.

Proof. By the above Remark, a flat $R$-module $F$ has the injective resolution $F \otimes_k V^*$. By Lemma 3.3, we complete the proof.

Theorem 3.5 ([Mi2]). Let $k$ be a commutative ring, and $R$ a left Noetherian projective $k$-algebra. Then $\bigoplus_{i=0}^n E^i$ is an injective cogenerator for the category of left $R$-modules of flat dimension $\leq n$.

Proof. Let $X$ be a left $R$-module of flat dimension $\leq n$, then $X$ has the following flat resolution:

$$0 \to F_n \to \ldots \to F_1 \to F_0 \to X \to 0.$$

By Lemma 3.4, the switch-back lemma, we get an injection from $X$ to $\bigoplus_{i=0}^n I_i$. Since $\bigoplus_{i=0}^n I_i$ belongs to $\text{Ps}(\bigoplus_{i=0}^n E^i)$, we complete the proof.

4. Applications to The Auslander Condition

Let $R$ be a coherent ring.

the Auslander condition for $R$ : for every finitely presented left $R$-module $M$ and every $i \geq 0$, every submodule $N$ of $\text{Ext}_R^i(M, R)$ satisfies that $\text{Ext}_R^j(N, R) = 0$ for all $0 \leq j < i$. \iff $\text{fdim}_RE^i \leq i$ for all $i \geq 0$ (see [FGR] for details).

$R$ : Auslander-Gorenstein $\iff$ a Noetherian ring $R$ satisfies the Auslander condition and that $\text{idim}_R, \text{idim}_R < \infty$

$R$ : Auslander regular $\iff$ $R$ is an Auslander-Gorenstein ring of which global dimension is finite.

Theorem 4.1 ([Mi2]). Let $k$ be a commutative ring, and $R$ a right coherent and left Noetherian projective $k$-algebra. The following are equivalent.

(a) $R$ satisfies the Auslander condition.

(b) $\text{fdim}_RE(M) \leq \text{fdim}_RM$ for every left $R$-module $M$. 
Proof. (a) $\Rightarrow$ (b): We may assume $\text{fdim}_R M = n < \infty$. According to Theorem 3.5, $M$ admits an injection to some injective left $R$-module $I$ in $\text{Ps}(\bigoplus_{i=0}^{n} E^i)$. By Lemma 3.2, we have $\text{fdim}_R I \leq n$. Since the injective hull $E(M)$ of $M$ is a direct summand of $I$, we have $\text{fdim}_R E(M) \leq n$.

(b) $\Rightarrow$ (a): By dimension shift, OK

Proposition 4.2 ([Mi2]). Let $k$ be a commutative ring, and $R$ a Noetherian projective $k$-algebra which satisfies the Auslander condition. Then $\bigoplus_{i=0}^{n} E^i$ is a $\Sigma$-embedding cogenerator for the category of left $R$-modules of flat dimension $\leq n$.

Proposition 4.3 ([Mi2]). Let $k$ be a commutative ring, and $R$ an Auslander-Gorenstein projective $k$-algebra of injective dimension $n$. For a left $R$-module $M$, the following are equivalent.

(a) $\text{fdim}_R M \leq m$.

(b) $\text{idim}_R M \leq n$ and $M$ has the following minimal injective resolution:

$$0 \to M \to I^0 \to I^1 \to I^2 \to \ldots,$$

where $\text{fdim}_R I^i \leq \min(i + m, n)$ for all $i \geq 0$.

Proof. (b) $\Rightarrow$ (a): Let $0 \to M \to I^0 \to I^1 \to I^2 \to \ldots \to I^s \to 0$ be a minimal injective resolution of $M$, where $s \leq n$. By dimension shifting, we have $\text{fdim}_R M \leq m$.

(a) $\Rightarrow$ (b): Let $0 \to F_m \to \ldots \to F_1 \to F_0 \to M \to 0$ be a flat resolution of $M$.

Every flat left $R$-module $F$ is of finite projective dimension, and then $\text{idim}_R F \leq n$.

By Lemma 3.4, it is easy to see that $F_i$ has the following injective resolution:

$$0 \to F_i \to I^{i0} \to I^{i1} \to \ldots \to I^{in} \to 0,$$

where $I^{i,n} \in \text{Ps}(\bigoplus_{i=0}^{n} E^i)$ for all $n \geq 0$. Then by the switch-back lemma, $M$ has the following injective resolution:

$$0 \to M \to Q^0 \to Q^1 \to \ldots \to Q^n \to 0,$$

where $Q^n$ is a direct summand of $\bigoplus_{i+j=0}^{\infty} I^{i,j}$, and where $Q^{k} = \bigoplus_{i+j=k}^{\infty} I^{i,j}$, ($k \geq 1$). Then, for every $0 \leq j \leq n$, we have $Q^j \in \text{Ps}(\bigoplus_{i+j=k}^{\min(j+m, n)} E^j)$. By the definition of Auslander-Gorenstein rings, we have $\text{fdim}_R Q^j \leq \min(j+m, n)$ for all $j \geq 0$. Since every $j$-th term of a minimal injective resolution of $M$ is a direct summand of $Q$, the condition (b) is satisfied.
Case $m+i < n$

Case $m+i = n$
Usual construction of an injective resolution of a complex (H. Cartan, S. Eilenberg or S. MacLane)

$L : a complex \ldots \rightarrow 0 \rightarrow L^0 \rightarrow L^1 \rightarrow L^2 \rightarrow \ldots$

Let $Z(L') \rightarrow J' \cdot, H'(L') \rightarrow K' \cdot$ be injective resolutions. Then we have injective resolutions $B'(L') \rightarrow M' \cdot, L' \rightarrow I' \cdot$ such that $0 \rightarrow J' \cdot \rightarrow M' \cdot \rightarrow K' \cdot \rightarrow 0$ and $0 \rightarrow M' \cdot \rightarrow I' \cdot \rightarrow J'_{n+1} \cdot \rightarrow 0$ are exact. Therefore we have the following double complex:

![Double Complex Diagram]

Mapping cones

$$\begin{align*}
  y_{n-1} & \rightarrow y_{n-2} \rightarrow y_{n-1} \rightarrow y_n \rightarrow \ldots \\
  M' & \rightarrow X_{n-1} \oplus Y_{n-2} \rightarrow X_n \oplus Y_{n-1} \rightarrow X_{n+1} \oplus Y_n \rightarrow \ldots \\
  X' & \rightarrow X_{n-1} \rightarrow X_n \rightarrow X_{n+1} \rightarrow \ldots \\
  Y' & \rightarrow y_{n-1} \rightarrow y_n \rightarrow \ldots
\end{align*}$$

where $\partial^n = \begin{pmatrix} d^n & 0 \\ fn & -\delta^n \end{pmatrix}$. Then the above morphisms of complexes induce the long exact sequence: $\ldots \rightarrow H^{n-1}(Y') \rightarrow H^n(M') \rightarrow H^n(X') \rightarrow H^n(Y') \rightarrow H^{n+1}(M') \rightarrow \ldots$
REFERENCES