REPRESENTATIONS AND QUIVERS FOR RING THEORISTS

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Contents

1. Modules and Representations

Throughout this note, k is a field, and we deal with associative k-algebras. A k-algebra A is a k-vector space with a k-bilinear map $\mu : A \times A \to A$ satisfying

$$(1.1) \qquad \begin{cases} 1_A \in A \\ \mu(1_A, a) = a \ ({}^{\forall}a \in A) \\ \mu(a, 1_A) = a \ ({}^{\forall}a \in A) \\ \mu \circ (\mu \times \mathbf{1}) = \mu \circ (\mathbf{1} \times \mu) \end{cases}$$

$$\begin{array}{ccc} A \times A \times A & \xrightarrow{\mu \times \mathbf{1}} & A \times A \\ {\scriptstyle \mathbf{1} \times \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ A \times A & \xrightarrow{\mu} & A \end{array}$$

In this note, for a k-algebra A, we fix a complete set $\{e_i | 1 \leq i \leq n\}$ of orthogonal primitive idempotents of A. Then we have

$$A = \bigoplus_{1 \leq i,j \leq n} e_i A e_j$$

as a k-vector space and a family of k-bilinear maps

$$\mu_{ijk} : e_i A e_j \times e_j A e_k \to e_i A e_k$$

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such that

$$
(1.2)\qquad \begin{cases} e_i \in e_iAe_i(^\forall i) \\ \mu_{iij}(e_i, a_{ij}) = a_{ij} \ (^\forall a_{ij} \in e_iAe_j) \\ \mu_{ijj}(a_{ij}, e_j) = a_{ij} \ (^\forall a_{ij} \in e_iAe_j) \\ \mu_{ikl} \circ (\mu_{ijk} \times \mathbf{1}) = \mu_{ijl} \circ (\mathbf{1} \times \mu_{jkl}) \end{cases}
$$

$$
\begin{array}{ccc} e_iAe_j \times e_jAe_k \times e_kAe_l & \xrightarrow{\mu_{ijk}\times\mathbf{1}} & e_iAe_k \times e_kAe_l \\ {\scriptstyle \mathbf{1}\times\mu_{jkl}}\downarrow & & \downarrow{\scriptstyle \mu_{ikl}} \\ e_iAe_j \times e_jAe_l & \xrightarrow{\mu_{ijl}} & e_iAe_l \end{array}
$$

Conversely, a system $(e_iAe_j\ (1 \leq i, j \leq n); \mu_{ijk}\ (1 \leq i, j, k \leq n))$ of k-vector spaces satisfying the equation 1.2 defines a k-algebra $A = \bigoplus_{1\leq i,j\leq n} e_iAe_j$ (in this case we define the other multiplications to be 0).

A (left) A-module M is a k-vector space with a k-bilinear map $\phi^M : A \times M \to M$ satisfying

$$
(1.3)\qquad \begin{cases} \phi^M(1_A, m) = m \ (^\forall m \in M) \\ \phi^M \circ (\mathbf{1} \times \phi^M) = \phi^M \circ (\mu \times \mathbf{1}) \end{cases}
$$

$$
\begin{array}{ccc} A \times A \times M & \xrightarrow{\mu\times\mathbf{1}} & A \times M \\ {\scriptstyle \mathbf{1}\times\phi^M}\downarrow & & \downarrow{\scriptstyle \phi^M} \\ A \times M & \xrightarrow{\phi^M} & M \end{array}
$$

As an equivalent notion, a representation M of A is a k-vector space with a k-algebra map $\psi : A \to \mathrm{End}_k(M)$, where $\mathrm{End}_k(M)$ is the k-vector space of k-linear endomaps of M.

For a complete set $\{e_i | 1 \leq i \leq n\}$ of orthogonal primitive idempotents of A, we have

$$
M = \bigoplus_{1\leq i\leq n} e_iM
$$

as a k-vector space and a family of k-bilinear maps

$$
\phi^M_{ji} : e_jAe_i \times e_iM \to e_jM
$$

such that

$$
(1.4)\qquad \begin{cases} \phi^M_{ii}(e_i, m_i) = m_i \ (^\forall m_i \in e_iM) \\ \phi^M_{kj} \circ (\mathbf{1} \times \phi^M_{ji}) = \phi^M_{ki} \circ (\mu_{kji} \times \mathbf{1}) \end{cases}
$$

$$
\begin{array}{ccc} e_kAe_j \times e_jAe_i \times e_iM & \xrightarrow{\mu_{kji}\times\mathbf{1}} & e_kAe_i \times e_iM \\ {\scriptstyle \mathbf{1}\times\phi^M_{ji}}\downarrow & & \downarrow{\scriptstyle \phi^M_{ki}} \\ e_kAe_j \times e_jM & \xrightarrow{\phi^M_{kj}} & e_kM \end{array}
$$

As an equivalent notion, a family $(e_iM)_{1\leq i\leq n}$ of k-vector spaces with k-linear maps $\psi^M_{ij} : e_iAe_j \to \mathrm{Hom}_k(e_jM, e_iM)$ such that

$$\left\{\begin{array}{l} \psi^M_{ii}(e_i) = \mathbf{1}_{e_iM} \\ \psi^M_{ik}(a_{ij}b_{jk}) = \psi^M_{ij}(a_{ij}) \circ \psi^M_{jk}(b_{jk}) \\ \qquad ({}^\forall a_{ij} \in e_iAe_j, {}^\forall b_{jk} \in e_jAe_k) \end{array}\right. \tag{1.5}$$

$$\begin{array}{ccc} e_kM & & \\ {\scriptstyle \psi^M_{jk}(b_{jk})}\Big\downarrow & \searrow{\scriptstyle \psi^M_{ik}(a_{ij}b_{jk})} & \\ e_jM & \xrightarrow[\psi^M_{ij}(a_{ij})]{} & e_iM \end{array}$$

where $\mathrm{Hom}_k(e_jM, e_iM)$ is the k-vector space of k-linear maps from e_jM to e_iM.

Example 1.6. For a left A-module Ae_r, a family $(e_iAe_r)_{1\leq i\leq n}$ with k-linear maps $\psi^{Ae_r}_{ij} : e_iAe_j \to \mathrm{Hom}_k(e_jAe_r, e_iAe_r)$ defined by $\psi^{Ae_r}_{ij}(a_{ij}) = \mu_{ijr}(a_{ij}, -)$.

$$e_jAe_r \xrightarrow{\mu_{ijr}(a_{ij},-)} e_iAe_r \quad (a_{ij} \in e_iAe_j)$$

For representations M, N, an A-homomorphism $f : M \to N$ is a k-linear map satisfying

$$f \circ \psi^M(a) = \phi^N(a) \circ f \ ({}^\forall a \in A) \tag{1.7}$$

$$\begin{array}{ccc} M & \xrightarrow{\psi^M(a)} & M \\ {\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ N & \xrightarrow[\psi^N(a)]{} & N \end{array}$$

Then we have a family $(f_i : e_iM \to e_iN)_{1\leq i\leq n}$ of k-linear maps satisfying

$$f_i \circ \psi^M_{ij}(a_{ij}) = \phi^N_{ij}(a_{ij}) \circ f_j \ ({}^\forall a_{ij} \in e_iAe_j) \tag{1.8}$$

$$\begin{array}{ccc} e_jM & \xrightarrow{\psi^M_{ij}(a_{ij})} & e_iM \\ {\scriptstyle f_j}\Big\downarrow & & \Big\downarrow{\scriptstyle f_i} \\ e_jN & \xrightarrow[\phi^N_{ij}(a_{ij})]{} & e_iN \end{array}$$

Conversely, it is easy to see that a system $(e_iM\ (1 \leq i \leq n); \psi^M_{ij}\ (1 \leq i, j \leq n))$ of k-vector spaces defines a left A-module $M = \bigoplus_{1\leq i\leq n} e_iM$ (in this case we define the other actions to be 0), and that a family $(f_i)_{1\leq i\leq n}$ of k-linear maps defines an A-homomorphism from M to N.

Example 1.9. For idempotents e_r, e_s of A, an A-homomorphism $\mu(-, b_{sr}) : Ae_s \to Ae_r$ is obtained by a family of k-linear maps $\mu_{isr}(-, b_{sr}) : e_iAe_s \to e_iAe_r$ $(1 \leq i \leq$

n).

$$\begin{array}{ccc} e_jAe_s & \xrightarrow{\mu_{ijs}(a_{ij},-)} & e_iAe_s \\ {\scriptstyle \mu_{jsr}(-,b_{sr})}\downarrow & & \downarrow{\scriptstyle \mu_{isr}(-,b_{sr})} \\ e_jAe_r & \xrightarrow{\mu_{ijr}(a_{ij},-)} & e_iAe_r \end{array}$$

Theorem 1.10. *Let* $\mathsf{Rep}\,A$ *be the category consisting of* $M = (M(i)\ (1 \le i \le n); \psi^M_{ij}(1 \le i, j \le n))$ *satisfying*

$$\begin{aligned} \psi^M_{ii}(e_i) &= \mathbf{1}_{M(i)} \\ \psi^M_{ik}(a_{ij}b_{jk}) &= \psi^M_{ij}(a_{ij}) \circ \psi^M_{jk}(b_{jk}) \\ &\quad ({}^{\forall}a_{ij} \in e_iAe_j, {}^{\forall}b_{jk} \in e_jAe_k) \end{aligned}$$

$$\begin{array}{ccc} M(k) & & \\ {\scriptstyle \psi^M_{jk}(b_{jk})}\downarrow & \searrow^{\psi^M_{ik}(a_{ij}b_{jk})} & \\ M(j) & \xrightarrow[\psi^M_{ij}(a_{ij})]{} & e_iM(i) \end{array}$$

as objects, and of $(f_i : M(i) \to N(i))_{1 \le i \le n}$ *satisfying*

$$f_i \circ \psi^M_{ij}(a_{ij}) = \phi^N_{ij}(a_{ij}) \circ f_j$$

$$\begin{array}{ccc} M(j) & \xrightarrow{\psi^M_{ij}(a_{ij})} & M(i) \\ {\scriptstyle f_j}\downarrow & & \downarrow{\scriptstyle f_i} \\ N(j) & \xrightarrow[\phi^N_{ij}(a_{ij})]{} & N(i) \end{array}$$

for M, N *as morphisms. Then* $\mathsf{Rep}\,A$ *is equivalent to the category* $\mathsf{Mod}\,A$ *of left* A*-modules.*

For A-modules M, N, we denote by $\mathrm{Hom}_A(M, N)$ the set of A-homomorphisms from M to N.

Lemma 1.11. *For a left* A*-module* M*, we have*

$$\mathrm{Hom}_A(Ae_i, M) \cong e_iM$$

as e_iAe_i*-modules.*

Proof. Let $\theta : \mathrm{Hom}_A(Ae_i, M) \to e_iM$ be the map defined by $(f) = f(e_i)$ for $f \in \mathrm{Hom}_A(Ae_i, M)$, and $\eta : e_iM \to \mathrm{Hom}_A(Ae_i, M)$ the map defined by $\eta(m_i)(ae_i) = am_i$ for $m_i \in e_iM$ and $ae_i \in Ae_i$. Then θ, η are A-homomorphisms and $\theta\eta = 1$, $\eta\theta = 1$. □

Corollary 1.12. *Let* J *be the Jacobson radical of* A*. Assume that* A *is a basic artinian* k*-algebra, that is,* $Ae_i \not\cong Ae_j$ *for* $i \ne j$*. Then we have*

$$\mathrm{Hom}_A(Ae_i, Ae_j/Je_j) \cong \begin{cases} e_iAe_i/e_iJe_i & \text{if } i = j \\ O & \text{if } i \ne j \end{cases}$$

Proposition 1.13. *Assume that A is a finite dimensional k-algebra satisfying $A/J \cong k \times \cdots \times k$ (i.e., $e_iAe_i/e_iJe_i \cong k$ for any $1 \leq i \leq n$). For a left A-module M, we have*

$$\dim_k e_iM = \text{ the appearance number of simple type } Ae_i/Je_i \text{ in a composition series of } M.$$

Proof. Let

$$O = M_{-1} \subset M_0 \subset M_1 \subset \cdots \subset M_r = M$$

be a composition series. Then we have an exact sequence

$$O \to \mathrm{Hom}_A(Ae_i, M_{t-1}) \to \mathrm{Hom}_A(Ae_i, M_t) \to \mathrm{Hom}_A(Ae_i, M_t/M_{t-1}) \to O$$

for $1 \leq t \leq r$. Therefore we have

$$\dim_k e_iM = \sum_{0 \leq t \leq r} \dim_k \mathrm{Hom}_A(Ae_i, M_t/M_{t-1}).$$

By Corollary 1.12, we get the statement. □

Example 1.14. In the case of $A/J \cong k \times \cdots \times k$, we may assume that $A = (\oplus_{i=1}^n ke_i) \oplus J$. A simple left A-module Ae_r/Je_r is described by $(M(i); \psi_{ij}^M) \in \mathsf{Rep}\, A$ as follows.

$$M(i) = \begin{cases} k & \text{if } i = r \\ O & \text{if } i \neq r \end{cases}$$

$$\psi_{ij}^M(a_{ij}) = \begin{cases} \lambda & \text{if } (i,j) = (r,r), a_{ij} = \lambda e_r \\ 0 & \text{if } (i,j) = (r,r), a_{ij} \in e_rJe_r \\ 0 & \text{otherwise} \end{cases}$$

2. Quivers and Path Algebras

Definition 2.1. A quiver $\vec{Q} = (Q_0, Q_1)$ is an oriented graph, where Q_0 is a set of vertices and Q_1 is a set of arrows between vertices. We use $h : Q_1 \to Q_0$, $t : Q_1 \to Q_0$ the maps defined by $h(\alpha) = j$, $t(\alpha) = i$ when $\alpha : i \to j$ is arrow from the vertex i to the vertex j. A quiver $\vec{Q} = (Q_0, Q_1)$ is called a finite quiver if $\#Q_0, \#Q_1 < \infty$.

A path $w = (i|\alpha_r, \ldots, \alpha_1|j)$ from the vertex j to the vertex i in the quiver $\vec{Q}$ is a sequence of ordered arrows $\alpha_1, \ldots, \alpha_r$ such that $j = t(\alpha_1), h(\alpha_i) = t(\alpha_{i+1})$ $(1 \leq i \leq r-1), h(\alpha_r) = i$. In this case, j (resp., i) is called the tail $t(w)$ (resp., the head $h(w)$) of w, and r is called the length of a path w. For every vertex i, the path $e_i = (i|\ |i)$ of length 0 is called the empty path. A non-empty path w is called an oriented cycle if $h(w) = t(w)$.

Definition 2.2. Let $Q_0 = \{1, \ldots, n\}$ and Q_1 a set. For any $i, j \in Q_0$, $e_i|\vec{Q}|e_j$ is the set of paths w in $\vec{Q}$ with $t(w) = j, h(w) = i$. For any $i, j, k \in Q_0$ with $e_i|\vec{Q}|e_j \neq \phi$, $e_j|\vec{Q}|e_k \neq \phi$, we define a composition map $\mu_{ijk} : e_i|\vec{Q}|e_j \times e_j|\vec{Q}|e_k \to e_i|\vec{Q}|e_k$ by setting

$$\mu_{ijk}((i|\alpha_s, \ldots, \alpha_{r+1}|j), (j|\alpha_r, \ldots, \alpha_1|k)) = (i|\alpha_s, \ldots, \alpha_1|k).$$

Then for any $i, j, k, l \in Q_0$ with $e_i|\vec{Q}|e_j \neq \phi$, $e_j|\vec{Q}|e_k \neq \phi$, $e_k|\vec{Q}|e_l \neq \phi$, we have

$$\begin{array}{ccc} e_i|\vec{Q}|e_j \times e_j|\vec{Q}|e_k \times e_k|\vec{Q}|e_l & \xrightarrow{\mu_{ijk}\times\mathbf{1}} & e_i|\vec{Q}|e_k \times e_k|\vec{Q}|e_l \\ {\scriptstyle \mathbf{1}\times\mu_{jkl}}\downarrow & & \downarrow{\scriptstyle \mu_{ikl}} \\ e_i|\vec{Q}|e_j \times e_j|\vec{Q}|e_l & \xrightarrow{\mu_{ijl}} & e_i|\vec{Q}|e_l \end{array}$$

We denote by $e_ik\vec{Q}e_j$ the k-vector space with the paths from the vertex j to i as a basis if $e_i|\vec{Q}|e_j \neq \phi$, and $e_ik\vec{Q}e_j = O$ if $e_i|\vec{Q}|e_j = \phi$. For any $i, j, k \in Q_0$, we define a k-bilinear map $\mu_{ijk} : e_ik\vec{Q}e_j \times e_jk\vec{Q}e_k \to e_ik\vec{Q}e_k$ by setting

$$\mu_{ijk}(\lambda_v v, \lambda_w w) = \lambda_v\lambda_w vw$$

with $\lambda_v, \lambda_w \in k$. For any $i, j, k, l \in Q_0$, we have

$$\begin{array}{ccc} e_ik\vec{Q}e_j \times e_jk\vec{Q}e_k \times e_kk\vec{Q}e_l & \xrightarrow{\mu_{ijk}\times\mathbf{1}} & e_ik\vec{Q}e_k \times e_kk\vec{Q}e_l \\ {\scriptstyle \mathbf{1}\times\mu_{jkl}}\downarrow & & \downarrow{\scriptstyle \mu_{ikl}} \\ e_ik\vec{Q}e_j \times e_jk\vec{Q}e_l & \xrightarrow{\mu_{ijl}} & e_ik\vec{Q}e_l \end{array}$$

Then, by 1.2, $k\vec{Q} = \bigoplus_{1\leq i,j\leq n} e_ik\vec{Q}e_j$ becomes an associative k-algebra. This algebra is called the path algebra of $\vec{Q}$ over k.

We often simply write $\alpha_r, \ldots, \alpha_1$ for $(i|\alpha_r, \ldots, \alpha_1|j)$.

Proposition 2.3. *For a finite quiver $\vec{Q}$, $k\vec{Q}$ is a finite dimensional k-algebra if and only if $\vec{Q}$ has no oriented cycle.*

Example 2.4. For a quiver

$$\vec{Q}:\ 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

we have

$$\begin{array}{lll} e_1k\vec{Q}e_1 =< e_1 >_k & e_2k\vec{Q}e_1 =< \alpha >_k & e_3k\vec{Q}e_1 =< \beta\alpha >_k \\ e_1k\vec{Q}e_2 = O & e_2k\vec{Q}e_2 =< e_2 >_k & e_3k\vec{Q}e_2 =< \beta >_k \\ e_1k\vec{Q}e_3 = O & e_2k\vec{Q}e_3 = O & e_3k\vec{Q}e_3 =< e_3 >_k \end{array}$$

Then we have

$$k\vec{Q} \cong \begin{bmatrix} k & 0 & 0 \\ k & k & 0 \\ k & k & k \end{bmatrix}$$

Example 2.5. For a quiver

$$\vec{Q}:\ 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2 \xrightarrow{\gamma} 3$$

we have

$$\begin{array}{lll} e_1k\vec{Q}e_1 =< e_1 >_k & e_2k\vec{Q}e_1 =< \alpha, \beta >_k & e_3k\vec{Q}e_1 =< \gamma\alpha, \gamma\beta >_k \\ e_1k\vec{Q}e_2 = O & e_2k\vec{Q}e_2 =< e_2 >_k & e_3k\vec{Q}e_2 =< \gamma >_k \\ e_1k\vec{Q}e_3 = O & e_2k\vec{Q}e_3 = O & e_3k\vec{Q}e_3 =< e_3 >_k \end{array}$$

Then we have

$$k\vec{Q} \cong \begin{bmatrix} k & O \\ {}_AM_k & A \end{bmatrix}, \quad A = \begin{bmatrix} k & 0 \\ k & k \end{bmatrix}, \quad {}_AM = \begin{bmatrix} k \\ k \end{bmatrix} \oplus \begin{bmatrix} k \\ k \end{bmatrix}$$

Example 2.6. For a quiver

$$\vec{Q}: \ 1 \xrightarrow{\alpha} 2 \circlearrowright \beta$$

we have

$$\begin{array}{ll} e_1 k\vec{Q} e_1 =< e_1 >_k & e_2 k\vec{Q} e_1 =< \alpha, \beta^n\alpha : n \in \mathbb{N} >_k \\ e_1 k\vec{Q} e_2 = O & e_2 k\vec{Q} e_2 =< e_2, \beta^n : n \in \mathbb{N} >_k \end{array}$$

Then we have

$$k\vec{Q} \cong \begin{bmatrix} k & 0 \\ k[x] & k[x] \end{bmatrix}$$

Lemma 2.7. *Let A be a ring, $O \to X \to Y \to Z \to O$ an exact sequence of left A-modules. Then we have*

$$\operatorname{pdim}_A Y \leq \max\{\operatorname{pdim}_A X, \operatorname{pdim}_A Z\}$$

Proposition 2.8. *For a finite dimensional k-algebra A, the following are equivalent.*

1. $\operatorname{lgldim} A \leq n$.
2. $\operatorname{pdim}_A A/J \leq n$.

In particular, the following are equivalent.

1. *A is hereditary.*
2. *J is projective.*
3. $\operatorname{lgldim} A \leq 1$.
4. $\operatorname{pdim}_A A/J \leq 1$.

Proposition 2.9. *Let $\vec{Q}$ be a finite quiver without oriented cycles. Then $k\vec{Q}$ is hereditary, and $k\vec{Q}/J_{k\vec{Q}} \cong k \times \cdots \times k$, where $J_{k\vec{Q}}$ is the Jacobson radical of $k\vec{Q}$.*

Proof. Let $Q_0 = 1, \ldots, n$, then $1 = e_1 + \ldots + e_n$. Let J_+ be the vector space spanned by paths of length ≥ 1, then there exists $t \geq 0$ such that $J_+^{t+1} = 0$. Therefore $J_+ \subset J_{k\vec{Q}}$. It is easy to see that $k\vec{Q}/J_+ \cong ke_1 \times \cdots \times ke_n$ as rings. Thus we have $J_+ = J_{k\vec{Q}}$. For $i \in Q_0$, since $\vec{Q}$ is finite, we may assume that the set of arrows α with $t(\alpha) = i$ is $\{\alpha_1, \ldots, \alpha_r\}$. Then we have

$$J_+ e_i = \oplus_{i=1}^r k\vec{Q}\alpha_i.$$

Since $\mu(-, \alpha_j) : k\vec{Q}e_{h(\alpha_i)} \to k\vec{Q}$ is an isomorphism, $J_{k\vec{Q}}e_i$ is a projective left $k\vec{Q}$-module, and hence $J_{k\vec{Q}}$ is a projective left $k\vec{Q}$-module. □

Definition 2.10. Given a quiver $\vec{Q} = (Q_0, Q_1)$, a representation $M = (M(i); \psi^M)$ of $\vec{Q}$ over a field k is a family $(M(i))_{i \in Q_0}$ of k-vector spaces together with a family $(\psi^M(\alpha) : M(j) \to M(i))_{j \xrightarrow{\alpha} i \in Q_1}$ of k-linear maps. A representation $M = (M(i); \psi^M)$ is called a finite dimensional representation if $M(i)$ is a finite dimensional k-vector space for every $i \in Q_0$.

For $(M(i);\psi^M), (N(i);\psi^N)$, a morphism $f:(M(i);\psi^M)\to(N(i);\psi^N)$ is a family $(f_i : M(i)\to N(i))_{i\in Q_0}$ of k-linear maps satisfying that we have a commutative diagram

$$\begin{array}{ccc} M(j) & \xrightarrow{\psi^M(\alpha)} & M(i) \\ {\scriptstyle f_j}\downarrow & & \downarrow{\scriptstyle f_i} \\ N(j) & \xrightarrow{\psi^N(\alpha)} & N(i) \end{array}$$

for any $j \xrightarrow{\alpha} i \in Q_1$.

We denote by $\mathsf{Rep}_k\,\vec{Q}$ (resp., $\mathsf{rep}_k\,\vec{Q}$) the category of representations (resp., finite dimensional representations) of $\vec{Q}$ over k.

Theorem 2.11. *For a finite quiver $\vec{Q}$, $\mathsf{Rep}_k\,\vec{Q}$ is equivalent to $\mathsf{Rep}\,k\vec{Q}$, and hence it is equivalent to $\mathsf{Mod}\,k\vec{Q}$. Moreover, $\mathsf{rep}_k\,\vec{Q}$ is equivalent to the category $\mathsf{mod}_{\mathrm{fd}}\,k\vec{Q}$ of finite dimensional left $k\vec{Q}$-modules.*

Sketch of The Proof. For any idempotents e_i, e_j of $k\vec{Q}$, all elements of $e_i k\vec{Q} e_j$ are k-linear combinations of paths from j to i. Then it is easy. □

Proposition 2.12. *For any collection $\{(M_\lambda;\psi^{M_\lambda})\}_{\lambda\in\Lambda}$ of representations of $\vec{Q}$ over k, $(\bigoplus_{\lambda\in\Lambda} M_\lambda; \bigoplus_{\lambda\in\Lambda}\psi^{M_\lambda})$ (resp., $(\prod_{\lambda\in\Lambda} M_\lambda; \prod_{\lambda\in\Lambda}\psi^{M_\lambda})$) is the direct sum (resp., the direct product) of $\{(M_\lambda;\psi^{M_\lambda})\}_{\lambda\in\Lambda}$.*

Example 2.13. For a quiver

$$\vec{Q}:\ 1 \xrightarrow{\ \alpha\ } 2 \xrightarrow{\ \beta\ } 3$$

$k\vec{Q} = < e_1, e_2, e_3, \alpha, \beta, \beta\alpha >_k$. A representation M of $\vec{Q}$ over k is the following

$$M(1) \xrightarrow{\psi^M(\alpha)} M(2) \xrightarrow{\psi^M(\beta)} M(3)$$

Then we define $M = M(1)\oplus M(2)\oplus M(3)$ to be a left A-module as follows. For $m = (m_1, m_2, m_3)\in M$ and $a = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_\alpha\alpha + \lambda_\beta\beta + \lambda_{\beta\alpha}\beta\alpha \in k\vec{Q}$, we define

$$am = (\lambda_1 m_1, \lambda_2 m_2 + \lambda_\alpha\psi^M(\alpha)(m_1), \lambda_3 m_3 + \lambda_\beta\psi^M(\beta)(m_2) + \lambda_{\beta\alpha}\psi^M(\beta)\psi^M(\alpha)(m_1))$$

By the standard technique of linear algebra, all indecomposable representations are up to isomorphisms the following

$$\begin{array}{lll} M_1:\ O\to O\to k & M_2:\ O\to k\to k & M_3:\ k\to k\to k \\ M_4:\ O\to k\to O & M_5:\ k\to k\to O & M_6:\ k\to O\to O \end{array}$$

It is easy to see that we have composition series of M_3 and M_5

$$\begin{array}{ccccccc} M_1: & O & \longrightarrow & O & \longrightarrow & k \\ \downarrow & \downarrow & & \downarrow & & \downarrow \\ M_2: & O & \longrightarrow & k & \longrightarrow & k \\ \downarrow & \downarrow & & \downarrow & & \downarrow \\ M_3: & k & \longrightarrow & k & \longrightarrow & k \end{array} \qquad \begin{array}{cccccc} M_4: & O & \longrightarrow & k & \longrightarrow & O \\ \downarrow & \downarrow & & \downarrow & & \downarrow \\ M_5: & k & \longrightarrow & k & \longrightarrow & O \end{array}$$

Then $M_1 \cong k\vec{Q}e_3/Je_3, M_2/M_1 \cong k\vec{Q}e_2/Je_2, M_3/M_2 \cong k\vec{Q}e_1/Je_1$, and $M_4 \cong k\vec{Q}e_2/Je_2, M_5/M_4 \cong k\vec{Q}e_1/Je_1$, where J is the Jacobson radical of $k\vec{Q}$. Moreover, $k\vec{Q}e_1 \cong M_3, Je_3 \cong JM_3 \cong k\vec{Q}e_2 \cong M_2$ and $Je_2 \cong JM_2 \cong k\vec{Q}e_3 \cong M_1$.

We often write modules by using composition series

$$M_1: 3 \quad M_2: \begin{smallmatrix} 2 \\ |\beta \\ 3 \end{smallmatrix} \quad M_3: \begin{smallmatrix} 1 \\ |\alpha \\ 2 \\ |\beta \\ 3 \end{smallmatrix} \quad M_4: 2 \quad M_5: \begin{smallmatrix} 1 \\ |\alpha \\ 2 \end{smallmatrix} \quad M_6: 1$$

3. Quivers with Relations

Definition 3.1. A relation σ on a quiver $\vec{Q}$ over a field k is a k-linear combinations $\sigma = \sum_{t=1}^{r} \lambda_t w_t$, where w are paths from j to i, $\lambda_t \in k$. A pair $(\vec{Q}, \rho)$ is called a quiver with relations over k if $\rho = \{\sigma_1, \ldots, \sigma_s\}$ where σ_i is a relation for every i. We denote $k(\vec{Q}, \rho) = k\vec{Q}/ < \rho >$, where $< \rho >$ is the two-sided ideal of $k\vec{Q}$ generated by relations of ρ. We denote by J_+ the two-sided ideal of $k\vec{Q}$ generated by arrows.

Proposition 3.2. *Let $(\vec{Q}, \rho)$ be a finite quiver with relations over k. If there is t such that $J_+^t \subset < \rho > \subset J_+^2$, then $\overline{J}_+ = rad(k(\vec{Q}, \rho))$, where $\overline{J}_+$ is the image of J_+ in $k(\vec{Q}, \rho)$.*

Proof. Let $A = k(\vec{Q}, \rho)$ and $J = rad(k(\vec{Q}, \rho))$. Since $\overline{J}_+{}^t = O$, we have $\overline{J}_+ \subset J$. It is clearly that $A/\overline{J}_+ \cong k\vec{Q}/J_+$ is semi-simple. Then $(J + \overline{J}_+)/\overline{J}_+ = O$, and hence $\overline{J}_+ \subset J$. □

Definition 3.3. For a quiver with relations $(\vec{Q}, \rho)$ over k, $\mathsf{Rep}_k(\vec{Q}, \rho)$ (resp., $\mathsf{rep}_k(\vec{Q}, \rho)$) is the full subcategory of $\mathsf{Rep}_k \vec{Q}$ (resp., $\mathsf{rep}_k \vec{Q}$) consisting objects $M = (M(i); \psi^M)$ with $\psi^M(\sigma) = 0$ for any relation σ of ρ. Here $\psi^M(w) = \psi^M(\alpha_r) \ldots \psi^M(\alpha_1)$ for $w = \alpha_r \ldots \alpha_1$, and $\psi^M(\sigma) = \Sigma_t \lambda_t \psi^M(w_t)$ for $\sigma = \Sigma_t \lambda_t w_t$.

Theorem 3.4. *For a finite quiver with relations $(\vec{Q}, \rho)$ over k, $\mathsf{Rep}_k(\vec{Q}, \rho)$ (resp., $\mathsf{rep}_k(\vec{Q}, \rho)$) is equivalent to $\mathsf{Mod}\, k(\vec{Q}, \rho)$ (resp., $\mathsf{mod}_{\mathrm{fd}}\, k(\vec{Q}, \rho)$).*

Sketch. According to Theorem 2.11 and the explanations before the theorem, $\psi^M(\sigma) = 0$ means that $\sigma M = O$ when we consider $M = \bigoplus_{i \in Q_0} M(i)$ as a left $k\vec{Q}$-module. □

Definition 3.5. For a quiver $\vec{Q}$, the opposite quiver $\vec{Q}^{\mathrm{op}}$ is the quiver with all arrows reversed. For a quiver with relations $(\vec{Q}, \rho)$ over k, $(\vec{Q}^{\mathrm{op}}, \rho^{\mathrm{op}})$ is similarly defined. Then $k(\vec{Q}, \rho)^{\mathrm{op}} = k(\vec{Q}^{\mathrm{op}}, \rho^{\mathrm{op}})$.

Let $\mathrm{D} = \mathrm{Hom}_k(-, k)$. For a representation $M = (M(i); \psi^M) \in \mathsf{Rep}_k(\vec{Q}, \rho)$, $\mathrm{D}\, M = (\mathrm{D}\, M(i); \psi^{\mathrm{D}\, M})$, where $\psi^{\mathrm{D}\, M}(\alpha) = \mathrm{D}\, \psi^M(\alpha)$. Then $\mathrm{D}\, M$ is a representation of $(\vec{Q}^{\mathrm{op}}, \rho^{\mathrm{op}})$ over k.

Proposition 3.6. *For a quiver with relations $(\vec{Q}, \rho)$ over k, D induces a duality between $\mathsf{rep}_k(\vec{Q}, \rho)$ and $\mathsf{rep}_k(\vec{Q}^{\mathrm{op}}, \rho^{\mathrm{op}})$.*

Remark 3.7. For a k-algebra A, idempotents e_i, e_j and $a_{ij} \in e_i A e_j$, we have a left A-homomorphism $\mu(-, a_{ij}) : Ae_i \to Ae_j$. Then we have a commutative diagram in

$\mathsf{Mod}\, A^{\mathrm{op}}$

$$\begin{array}{ccc}
\mathrm{Hom}_A(Ae_j, A) & \xrightarrow{\mathrm{Hom}_A(\mu(-,a_{ij}),A)} & \mathrm{Hom}_A(Ae_i, A) \\
\wr\downarrow & & \downarrow\wr \\
e_jA & \xrightarrow[\mu(a_{ij},-)]{} & e_iA
\end{array}$$

In $\mathsf{Rep}_k\, \vec{Q}$, we have also the same result.

Example 3.8. For a quiver

$$\vec{Q}:\ 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

with a relation $\rho = \beta\alpha$. Then $k\vec{Q} =< e_1, e_2, e_3, \alpha, \beta, \beta\alpha >_k$ and the ideal $< \rho > =< \beta\alpha >_k$. Therefore $k(\vec{Q}, \rho) =< \overline{e}_1, \overline{e}_2, \overline{e}_3, \overline{\alpha}, \overline{\beta} >_k$. Let $A = k(\vec{Q}, \rho)$, then we have

$$\begin{array}{lll}
\overline{e}_1 A \overline{e}_1 =< \overline{e}_1 >_k & \overline{e}_2 A \overline{e}_1 =< \overline{\alpha} >_k & \overline{e}_3 A \overline{e}_1 = O \\
\overline{e}_1 A \overline{e}_2 = O & \overline{e}_2 A \overline{e}_2 =< \overline{e}_2 >_k & \overline{e}_3 A \overline{e}_2 =< \overline{\beta} >_k \\
\overline{e}_1 A \overline{e}_3 = O & \overline{e}_2 A \overline{e}_3 = O & \overline{e}_3 A \overline{e}_3 =< \overline{e}_3 >_k
\end{array}$$

Since this algebra is a factor of the path algebra in Example 2.13, all indecomposable representations are up to isomorphisms the following

$$\begin{array}{lll}
M_1:\ O \to O \to k & M_2:\ O \to k \to k & \\
M_4:\ O \to k \to O & M_5:\ k \to k \to O & M_6:\ k \to O \to O
\end{array}$$

The opposite quiver of with relations $(\vec{Q}^{\mathrm{op}}, \rho^{\mathrm{op}})$ is

$$\vec{Q}^{\mathrm{op}}:\ 1 \xleftarrow{\alpha^{\mathrm{op}}} 2 \xleftarrow{\beta^{\mathrm{op}}} 3$$

with $\rho^{\mathrm{op}} = \alpha^{\mathrm{op}}\beta^{\mathrm{op}}$. Therefore we have

$$\begin{array}{lll}
Ae_3 = Ae_3/Je_3 \cong M_1 & Ae_2 \cong \mathrm{D}(e_3A) \cong M_2 & \\
Ae_2/Je_2 \cong M_4 & Ae_1 \cong \mathrm{D}(e_2A) \cong M_5 & \mathrm{D}(e_1A) \cong Ae_1/Je_1 \cong M_6
\end{array}$$

$$M_1:\ 3 \quad M_2:\ \begin{smallmatrix} 2 \\ |\beta \\ 3 \end{smallmatrix} \quad M_4:\ 2 \quad M_5:\ \begin{smallmatrix} 1 \\ |\alpha \\ 2 \end{smallmatrix} \quad M_6:\ 1$$

Since projective resolutions of $Ae_1/Je_1, Ae_2/Je_2, Ae_3/Je_3$ are

$$\begin{array}{ccccccccccc}
O & \longrightarrow & Ae_3 & \longrightarrow & Ae_2 & \longrightarrow & Ae_1 & \longrightarrow & Ae_1/Je_1 & \longrightarrow & O \\
O & \longrightarrow & 3 & \longrightarrow & \begin{smallmatrix} 2 \\ |\beta \\ 3 \end{smallmatrix} & \longrightarrow & \begin{smallmatrix} 1 \\ |\alpha \\ 2 \end{smallmatrix} & \longrightarrow & 1 & \longrightarrow & O \\
O & \longrightarrow & Ae_3 & \longrightarrow & Ae_2 & \longrightarrow & Ae_2/Je_2 & \longrightarrow & O & & \\
O & \longrightarrow & 3 & \longrightarrow & \begin{smallmatrix} 2 \\ |\beta \\ 3 \end{smallmatrix} & \longrightarrow & 2 & \longrightarrow & O & & \\
O & \longrightarrow & Ae_3 & = & Ae_3 & \longrightarrow & O & & & & \\
O & \longrightarrow & 3 & = & 3 & \longrightarrow & O & & & &
\end{array}$$

by Proposition 2.8, $\mathrm{lgldim}\, k(\vec{Q}, \rho) = 2$. Moreover, an injective resolution of ${}_AA$ is

$$O \longrightarrow {}_AA \longrightarrow \mathrm{D}(e_2A) \oplus \mathrm{D}(e_3A)^2 \longrightarrow \mathrm{D}(e_2A) \longrightarrow \mathrm{D}(e_1A) \longrightarrow O$$

Since $\mathrm{pdim}_A \mathrm{D}(e_2A) = \mathrm{pdim}_A \mathrm{D}(e_3A) = 0$ and $\mathrm{pdim}_A \mathrm{D}(e_1A) = 2$, A is an Auslander regular k-algebra.

Example 3.9. For a quiver

$$\vec{Q}:\ 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

with a relation $\rho = \{\beta\alpha\}$. Then

$$k\vec{Q} = < e_1, e_2, \alpha, \beta, (\beta\alpha)^h, (\alpha\beta)^l\alpha(\beta\alpha)^m, \beta(\alpha\beta)^n : h, l, m, n \in \mathbb{N} >_k$$

and the ideal

$$< \rho > = < (\beta\alpha)^h, (\alpha\beta)^{l+1}, \alpha(\beta\alpha)^m, \beta(\alpha\beta)^n : h, l, m, n \in \mathbb{N} >_k .$$

Therefore $k(\vec{Q}, \rho) = < \overline{e}_1, \overline{e}_2, \overline{e}_3, \overline{\alpha}, \overline{\beta}, \overline{\alpha\beta} >_k$. Let $A = k(\vec{Q}, \rho)$, then we have

$$\begin{array}{ll} \overline{e}_1 A \overline{e}_1 = < \overline{e}_1 >_k & \overline{e}_2 A \overline{e}_1 = < \overline{\alpha} >_k \\ \overline{e}_1 A \overline{e}_2 = < \overline{\beta} >_k & \overline{e}_2 A \overline{e}_2 = < \overline{e}_2, \overline{\alpha\beta} >_k \end{array}$$

The opposite quiver of with relations $(\vec{Q}^{\mathrm{op}}, \rho^{\mathrm{op}})$ is

$$\vec{Q}^{\mathrm{op}}:\ 1 \underset{\beta^{\mathrm{op}}}{\overset{\alpha^{\mathrm{op}}}{\leftrightarrows}} 2$$

with a relation $\rho^{\mathrm{op}} = \{\alpha^{\mathrm{op}}\beta^{\mathrm{op}}\}$. Hence we have

$$Ae_1 : \begin{smallmatrix} 1 \\ |\alpha \\ 2 \end{smallmatrix} :\ k \underset{0}{\overset{1}{\rightleftarrows}} k \qquad Ae_2 \cong \mathrm{D}(e_2A) : \begin{smallmatrix} 2 \\ |\beta \\ 1 \\ |\alpha \\ 2 \end{smallmatrix} :\ k \underset{[\,1\ 0\,]}{\overset{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]}{\rightleftarrows}} k^2$$

$$\mathrm{D}(e_1A) : \begin{smallmatrix} 2 \\ |\beta \\ 1 \end{smallmatrix} :\ k \underset{1}{\overset{0}{\rightleftarrows}} k$$

$$Ae_1/Je_1 :\ 1 :\ k \underset{0}{\overset{0}{\rightleftarrows}} O \qquad Ae_2/Je_2 :\ 2 :\ O \underset{0}{\overset{0}{\rightleftarrows}} k$$

Since projective resolutions of $Ae_1/Je_1, Ae_2/Je_2$ are

$$\begin{array}{ccccccccccc} O & \longrightarrow & Ae_1 & \longrightarrow & Ae_2 & \longrightarrow & Ae_1 & \longrightarrow & Ae_1/Je_1 & \longrightarrow & O \\ O & \longrightarrow & \begin{smallmatrix} 1 \\ |\alpha \\ 2 \end{smallmatrix} & \longrightarrow & \begin{smallmatrix} 2 \\ |\beta \\ 1 \\ |\alpha \\ 2 \end{smallmatrix} & \longrightarrow & \begin{smallmatrix} 1 \\ |\alpha \\ 2 \end{smallmatrix} & \longrightarrow & 1 & \longrightarrow & O \\ O & \longrightarrow & Ae_1 & \longrightarrow & Ae_2 & \longrightarrow & Ae_2/Je_2 & \longrightarrow & O & & \\ O & \longrightarrow & \begin{smallmatrix} 1 \\ |\alpha \\ 2 \end{smallmatrix} & \longrightarrow & \begin{smallmatrix} 2 \\ |\beta \\ 1 \\ |\alpha \\ 2 \end{smallmatrix} & \longrightarrow & 2 & \longrightarrow & O & & \end{array}$$

by Proposition 2.8, $\operatorname{lgldim} A = 2$. A projective resolution of $\mathrm{D}(e_1A)$ is

$$\begin{array}{ccccccccccc} O & \longrightarrow & Ae_1 & \longrightarrow & Ae_2 & \longrightarrow & Ae_2 & \longrightarrow & \mathrm{D}(e_1A) & \longrightarrow & O \\ O & \longrightarrow & \begin{smallmatrix} 1 \\ |\alpha \\ 2 \end{smallmatrix} & \longrightarrow & \begin{smallmatrix} 2 \\ |\beta \\ 1 \\ |\alpha \\ 2 \end{smallmatrix} & \longrightarrow & \begin{smallmatrix} 2 \\ |\beta \\ 1 \\ |\alpha \\ 2 \end{smallmatrix} & \longrightarrow & \begin{smallmatrix} 2 \\ |\beta \\ 1 \end{smallmatrix} & \longrightarrow & O \end{array}$$

Moreover, an injective resolution of ${}_AA$ is

$$O \longrightarrow {}_AA \longrightarrow \mathrm{D}(e_2A)^2 \longrightarrow \mathrm{D}(e_2A) \longrightarrow \mathrm{D}(e_1A) \longrightarrow O$$

Since $\operatorname{pdim}_A \mathrm{D}(e_2A) = 0$ and $\operatorname{pdim}_A \mathrm{D}(e_1A) = 2$, A is an Auslander regular k-algebra.

Definition 3.10. Let Λ be a ring, and V a Λ-bimodule. We denote by $V^{\otimes n} = \overbrace{V \otimes_\Lambda \cdots \otimes_\Lambda V}^{n \text{ times}}$. Then the tensor ring $\mathrm{T}(\Lambda, V)$ is $\Lambda \oplus (\bigoplus_{n\geq 1} V^{\otimes n})$ as an abelian group, and its multiplication is induced by the canonical Λ-bilinear maps $V^{\otimes m} \otimes_\Lambda V^{\otimes n} \to V^{\otimes m+n}$ for $m, n \geq 0$.

Lemma 3.11. *Let Λ be a ring, V a Λ-bimodule and A a Λ-algebra. For a Λ-bimodule homomorphism $f : V \to A$, there exists a unique Λ-algebra homomorphism $\tilde{f} : \mathrm{T}(\Lambda, V) \to A$ such that $\tilde{f}|_V = f$.*

Sketch of The Proof. Let $\phi : \Lambda \to A$ be a ring homomorphism. A map $\tilde{f}$: $\mathrm{T}(\Lambda, V) \to A$ is defined by

$$\tilde{f}(a_0 + \sum_{i\geq 1}\sum_j v_{i1j} \otimes \cdots \otimes v_{iij}) = \phi(a_0) + \sum_{i\geq 1}\sum_j f(v_{i1j}) \dots f(v_{iij})$$

for $a_0 + \sum_{i\geq 1}\sum_j v_{i1j} \otimes \cdots \otimes v_{iij} \in \mathrm{T}(\Lambda, V)$. Then this satisfies the desired property. $\square$

Definition 3.12. For a k-algebra $\Lambda = \prod_{i=1}^n k$ and Λ-bimodule V, the quiver $\vec{Q}_{\mathrm{T}(\Lambda,V)}$ of $\mathrm{T}(\Lambda, V)$ consists of $Q_{\mathrm{T}(\Lambda,V)0} = \{1, \dots, n\}$, and of the number of arrows from the vertex i to j which is $\dim_k e_j V e_i$, where e_i, e_j correspond to i, j.

For a finite dimensional k-algebra A with $A/J_A \cong \prod_{i=1}^n k$, the quiver $\vec{Q}_A$ is the quiver $\vec{Q}_{\mathrm{T}(A/J_A, J_A/J_A^2)}$.

Proposition 3.13. *For a k-algebra $\Lambda = \prod_{i=1}^n k$ and Λ-bimodule V, there is a k-algebra isomorphism $\phi : \mathrm{T}(\Lambda, V) \to k\vec{Q}_{\mathrm{T}(\Lambda,V)}$.*

Proof. Since $k\vec{Q} = (\oplus_{i=1}^n \lambda_i e_i) \oplus J_+$, we identify the idempotents of A/J with them of $k\vec{Q}$. For $1 \leq i, j \leq n$, we take a k-basis $\{v_{ijk} | 1 \leq k \leq n_{ij}\}$ of $e_i V e_j$, and denote by $\alpha_{v_{ijk}}$ the arrow in $k\vec{Q}_{\mathrm{T}(\Lambda,V)}$ corresponding to v_{ijk}. A map $\phi : \mathrm{T}(\Lambda, V) \to A$ is defined by

$$\phi(\sum_{i=1}^n \lambda_i e_i + \sum_{i\geq 1, j} \lambda_{ij} u_{i1j} \otimes \cdots \otimes u_{iij}) = \sum_{i=1}^n \lambda_i e_i + \sum_{i\geq 1, j} \lambda_{ij} \alpha_{u_{i1j}} \dots \alpha_{u_{iij}}$$

for $\sum_{i=1}^n \lambda_i e_i + \sum_{i\geq 1,j} \lambda_{ij} u_{i1j} \otimes \cdots \otimes u_{iij} \in \mathrm{T}(\Lambda, V)$, where u_{ijk} are elements of the above basis. It is easy to see that $\dim_k e_i(\bigoplus_{n\geq 1} V^{\otimes n})e_j = e_i k\vec{Q} e_j$. Hence ϕ is bijective. $\square$

Theorem 3.14. *Let A be a finite dimensional k-algebra with $A/J_A \cong \prod_{i=1}^n k$. Then the following hold.*

1. *There is a surjective ring homomorphism $\phi : \mathrm{T}(A/J_A, J_A/J_A^2) \to A$ such that $\coprod_{i\geq \mathrm{rl}(A)} (J_A/J_A^2)^i \subset \operatorname{Ker} \phi \subset (J_A/J_A^2)^2$, where $\mathrm{rl}(A)$ is the Loewy length of A (i.e. $\mathrm{rl}\, A = \min\{t | J_A^{t+1} = 0\}$).*
2. *$A \cong k(\vec{Q}, \rho)$ with $J_A^r \subset <\rho> \subset J_A^2$ for some r, where $\vec{Q} = \vec{Q}_A$.*

Proof. 1. By the assumption, we may assume that a split injective k-algebra homomorphism $\phi_0 : A/J \to A$, $A/J = \oplus_{i=1}^n k e_i$ and $A = A/J \oplus J$ with $J = J_A$ the Jacobson radical of A. For any e_i, e_j, we choose elements $r_{ij1}, \dots, r_{ijn_{ij}}$ of $e_i J e_j$ such that $\{\overline{r}_{ij1}, \dots, \overline{r}_{ijn_{ij}}\}$ is a k-basis of $e_i(J/J^2)e_j$. Let $\phi_1 : J/J^2 \to A$ be an A/J-bimodule homomorphism defined by $\phi_1(\overline{r}_{ijk}) = r_{ijk}$, then by Lemma 3.11, there exists an

A/J-algebra homomorphism $\phi : \mathrm{T}(A/J, J/J^2) \to A$ such that $\tilde{\phi}|_{A/J\oplus J/J^2} = \phi_0 \oplus \phi_1$ is injective. Therefore $\coprod_{i\geq \mathrm{rl}(A)}(J_A/J_A^2)^i \subset \mathrm{Ker}\,\phi \subset (J_A/J_A^2)^2$, because of $J^{t+1} = 0$ for $t = \mathrm{rl}(A)$. If $\mathrm{rl}(A) = 1$, then ϕ is clearly bijective. In order to prove that ϕ is surjective, it suffices to show that for any $m \geq 1$ and any $x \in J^m$, there exists $y \in (\phi(J/J^2))^m$ such that $x - y \in J^{m+1}$. In the case of $m = 1$, it is trivial. In the case of $m \geq 2$, for $x \in J^m$ we have $x = \sum_i v_i w_i$, where $v_i \in J$ and $w_i \in J^{m-1}$. Then there are $y_i \in \phi(J/J^2)$ and $z_i \in (\phi(J/J^2))^{m-1}$ such that $v_i - y_i \in J^2$ and $w_i - z_i \in J^m$. Since $v_i \in J$ and $z_i \in J^{m-1}$, $v_i w_i - y_i z_i = v_i(w_i - z_i) + (v_i - y_i)z_i \in J^{m+1}$ and hence $x - \sum_i y_i z_i \in J^{m+1}$.

2. According to Proposition 3.13, we have a surjective k-algebra homomorphism $\phi : k\vec{Q} \to A$, where $\vec{Q} = \vec{Q}_A$. Let $t = \mathrm{rl}(A) + 1$, then ϕ induces a surjective k-algebra homomorphism $\psi : k\vec{Q}/J_+^t \to A$. Since $k\vec{Q}/J_+^t$ is a finite dimensional k-algebra, $\mathrm{Ker}\,\psi$ is a finitely generated ideal. Hence $\mathrm{Ker}\,\phi$ is a finitely generated ideal $< \sigma_1, \dots, \sigma_s >$ of $k\vec{Q}$, because J_+^t is a finitely generated ideal of $k\vec{Q}$. Since $\sigma_h = \sum_{ij} e_i \sigma_h e_j$, there is a set ρ of relations such that $\mathrm{Ker}\,\phi =< \rho >$. □

Lemma 3.15. *Let A be a hereditary finite dimensional k-algebra, I a two-sided ideal of A with $I \subset J_A^2$. Then A/I is not hereditary.*

Proof. Consider the exact sequence in $\mathsf{Mod}\,A/I$

$$O \to I/IJ_A \to J_A/IJ_A \xrightarrow{\pi} J_A/I \to O.$$

By Nakayama's Lemma, $I/IJ_A \neq O$. Since J_A is A-projective, J_A/IJ_A is A/I-projective. $I \subset J_A^2$ implies $I/IJ_A \subset J_A^2/IJ_A = J_{A/I}(J_A/IJ_A)$, If J_A/I is A/I-projective, then there is $\eta : J_A/I \to J_A/IJ_A$ such that $\pi\eta = 1_{J_A/I}$, and then $J_{A/I}(J_A/IJ_A) \oplus \mathrm{Im}\,\eta = J_A/IJ_A$. By Nakayama's Lemma, $\mathrm{Im}\,\eta = J_A/IJ_A$ and $I/IJ_A = O$. This is a contradiction. Hence J_A/I is not A/I-projective. By Proposition 2.8, we get the statement. □

Proposition 3.16. *Let A be a finite dimensional k-algebra with $A/J_A \cong k\times\cdots\times k$. Then the following are equivalent.*

1. *A is hereditary.*
2. *$A \cong k\vec{Q}_A$.*

Proof. 1 ⇒ 2. Let $f : Ae_i \to Ae_j$ be a non-zero A-homomorphism for primitive idempotents i, j. If f is not an isomorphism, then f is a monomorphism, because $\mathrm{Im}\,f$ is projective. Then there is no path $Ae_{i_1} \to \cdots \to Ae_{i_n} = Ae_{i_1}$ of non-zero A-homomorphisms which are not isomorphisms. Hence $\vec{Q}$ has no oriented cycle, $k\vec{Q}$ is a finite dimensional k- algebra. By Lemma 3.15, $A \cong k\vec{Q}_A$.

2 ⇒ 1. By Proposition 2.9, it is trivial. □

4. Base Extensions and Representations

Let k be a field and R a k-algebra. For a quiver with relations $(\vec{Q}, \rho)$ over a field k, let $e_1, \dots, e_n$ be the set of idempotents corresponding to vertices in $\vec{Q}$, $A = k(\vec{Q}, \rho)$ and $A^R = R\otimes_k k(\vec{Q}, \rho)$. Then we can consider that $A^R = \bigoplus_{\mathrm{path}\ w} R\overline{w}$ and $r\overline{w} = \overline{w}r$ for any $r \in R$ and any path w in $\vec{Q}$.

A left A^R-module M is a left A-module, and it is a direct sum $\bigoplus_{i=1}^n e_iM$ as an R-module. For any $\alpha \in Q_1$, we have

$$\begin{aligned}\alpha(rm) &= (\alpha r)m \\ &= (r\alpha)m \\ &= r(\alpha m)\end{aligned}$$

with $r \in R$, $m \in M$. Then $\psi^M(\alpha) : e_jM \to e_iM$ is a left R-linear map, and we get a system $(e_iM; \psi^M)$ of left R-modules satisfying

1. e_iM is a left R-module for any i.
2. $\psi^M(\alpha)$ is a left R-linear map for any $\alpha \in Q_1$.
3. $\psi^M(\sigma) = 0$ for any relation $\sigma \in \rho$.

For a left A^R-homomorphism $f : M \to N$, we get left R-linear maps $e_if = f_i : e_iM \to e_iN$ $(1 \le i \le n)$ such that

$$f_i \circ \psi^M(\alpha) = \psi^N(\alpha) \circ f_j \tag{4.1}$$

for any $\alpha \in Q_1$.

$$\begin{array}{ccc} e_jM & \xrightarrow{\psi^M(\alpha)} & e_iM \\ {\scriptstyle f_j}\downarrow & & \downarrow{\scriptstyle f_i} \\ e_jN & \xrightarrow[\psi^N(\alpha)]{} & e_iN \end{array}$$

Theorem 4.2. *Let $A = k(\vec{Q}, \rho)$, and let $\mathsf{Rep}_{R/k}(\vec{Q}, \rho)$ be the category consisting of $M = (M(i)\ (1 \le i \le n); \psi^M(\alpha)(\alpha \in Q_1))$ satisfying*

1. *$M(i)$ is a left R-module for any i.*
2. *$\psi^M(\alpha)$ is a left R-linear map for any $\alpha \in Q_1$.*
3. *$\psi^M(\sigma) = 0$ for any relation $\sigma \in \rho$.*

as objects, and of $(f_i : M(i) \to N(i))_{1 \le i \le n}$ satisfying

$$f_i \circ \psi^M(\alpha) = \phi^N(\alpha) \circ f_j$$

$$\begin{array}{ccc} M(j) & \xrightarrow{\psi^M(\alpha)} & M(i) \\ {\scriptstyle f_j}\downarrow & & \downarrow{\scriptstyle f_i} \\ N(j) & \xrightarrow[\phi^N(\alpha)]{} & N(i) \end{array}$$

for M, N as morphisms. Then $\mathsf{Rep}_{R/k}(\vec{Q}, \rho)$ is equivalent to the category $\mathsf{Mod}\, A^R$ of left A^R-modules.

Sketch of The Proof. By the above, we can construct a functor from $\mathsf{Mod}\, A^R$ to $\mathsf{Rep}_{R/k}(\vec{Q}, \rho)$. Conversely, given $M = (M(i); \psi^M) \in \mathsf{Rep}_{R/k}(\vec{Q}, \rho)$, let $M = \bigoplus_{i=1}^n M(i)$. For any $r \in R$, any arrow $\alpha : i \to j$ and $m \in M(i)$, we define the left A^R-action

$$(r\alpha)m = r\psi^M(\alpha)(m)$$

Then for any $r, s \in R$, any arrow $\alpha : i \to j, \beta : j \to l$ and $m \in M(i)$, we have

$$\begin{aligned}(s\beta)((r\alpha)m) &= (s\beta)(r\psi^M(\alpha)(m)) \\ &= (s\psi^M(\beta))(r\psi^M(\alpha)(m)) \\ &= s(r\psi^M(\beta)(\psi^M(\alpha)(m))) \\ &= sr(\psi^M(\beta)\psi^M(\alpha))(m)) \\ &= (sr\beta\alpha)(m)\end{aligned}$$

Therefore M becomes a left A^R-module. For a family $(f_i : M(i) \to N(i))_{1 \le i \le n}$ of morphisms , let $f = \oplus_{i=1}^n f_i$. For any $r \in R$, any arrow $\alpha : i \to j$ and $m \in M(i)$, we have

$$\begin{aligned}f_j(r\alpha m) &= f_j(r\psi^M(\alpha)(m)) \\ &= r(f_j \circ \psi^M(\alpha))(m) \\ &= r(\psi^N(\alpha) \circ f_i)(m) \\ &= (r\alpha)f_i(m)\end{aligned}$$

Hence f becomes a left A^R-homomorphism. It is easy to see that this construction defines a functor from $\mathsf{Rep}_{R/k}(\vec{Q}, \rho)$ to $\mathsf{Mod}\, A^R$, and it is an equivalence. $\square$

5. Examples related to Tachikawa's Conjecture

Conjecture 5.1 (Nakayama's Conjecture)**.** Let A be a finite dimensional algebra over a field k, and

$$O \to {}_AA \to I^0 \to I^1 \to \dots$$

an injective resolution of a left A-module ${}_AA$. If all I^i are projective, then A is self-injective.

Tachikawa showed that the above conjecture is equivalent to the pair of the following two conjectures.

Conjecture 5.2 (Tachikawa's Conjectures)**.** Let A be a finite dimensional algebra over a field k, M a finitely generated left A-module.

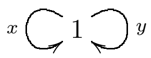
1. If A is self-injective and $\mathrm{Ext}_A^i(M, M) = O$ for all $i \ge 1$, then M is projective.
2. If $\mathrm{Ext}_A^i(\mathrm{D}\, A, A) = O$ for all $i \ge 1$, then A is self-injective.

R. Schultz showed that 1 of Conjecture 5.2 is not true in the case of A being an artinian ring [Sc]. I introduce his examples here.

5.1. The Case of Algebras.

For a quiver

$\vec{Q}$:



with relations $\rho = \{yx - \delta xy, x^2, y^2\}$ where $\delta \in k^\times$. Then

$$k\vec{Q} = \text{the free } k\text{-algebra } k < x, y >$$

and the ideal

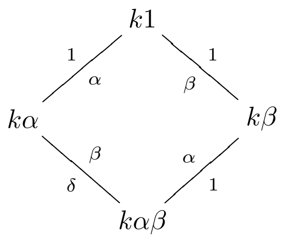
$$\begin{aligned}< \rho > = k < x, y > (yx - \delta xy)k < x, y > + \\ k < x, y > x^2 k < x, y > + k < x, y > y^2 k < x, y >\end{aligned}$$

Therefore $k(\vec{Q},\rho) =< 1, \alpha, \beta, \alpha\beta >_k$ is a local k-algebra, where $\alpha = \overline{x}, \beta = \overline{y}$. The multiplication of $k(\vec{Q},\rho)$ is

$$\begin{aligned}&(a1 + b_1\alpha + b_2\beta + c\alpha\beta)(a'1 + b_1'\alpha + b_2'\beta + c'\alpha\beta)\\ &= aa'1 + (ab_1' + a'b_1)\alpha + (ab_2' + a'b_1)\beta + (ac' + a'c + b_1b_2' + \delta b_2b_1')\alpha\beta\end{aligned}$$

with $a, b_1, b_2, c, a', b_1', b_2', c' \in k$. Then we have

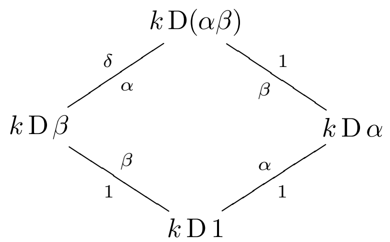
${}_AA:$



$$\begin{bmatrix} 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&1&0 \end{bmatrix} \circlearrowright k^4 \circlearrowleft \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&\delta&0&0 \end{bmatrix}$$

Since it is easy to see that A has the simple socle, A is self-injective. Indeed, $\mathrm{D}\,A =< \mathrm{D}\,1, \mathrm{D}\,\alpha, \mathrm{D}\,\beta, \mathrm{D}(\alpha\beta) >_k$

${}_A\mathrm{D}\,A:$



$$\begin{bmatrix} 0&0&0&0\\ \delta&0&0&0\\ 0&0&0&0\\ 0&0&1&0 \end{bmatrix} \circlearrowright k^4 \circlearrowleft \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&1&0&0 \end{bmatrix}$$

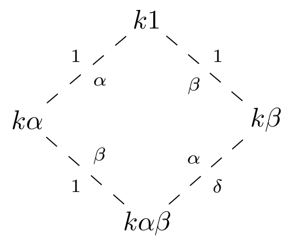
(We calculate the action as follows. $(\alpha\,\mathrm{D}(\alpha\beta))(\beta) = \mathrm{D}(\alpha\beta)(\beta\alpha) = \mathrm{D}(\alpha\beta)(\delta\alpha\beta)$ $= \delta$ implies $\alpha\,\mathrm{D}(\alpha\beta) = \delta\,\mathrm{D}\,\beta$). Then every isomorphism from ${}_AA$ to ${}_A\mathrm{D}\,A$ is the form $\begin{bmatrix} a&0&0&0\\ b&\delta a&0&0\\ c&0&a&0\\ d&c&b&a \end{bmatrix}$ with $a \in k^\times$.

On the other hand, the opposite quiver with relations $(\vec{Q}^{\mathrm{op}}, \rho^{\mathrm{op}})$ is

$$\vec{Q}^{\mathrm{op}} : x^{\mathrm{op}} \circlearrowright 1 \circlearrowleft y^{\mathrm{op}}$$

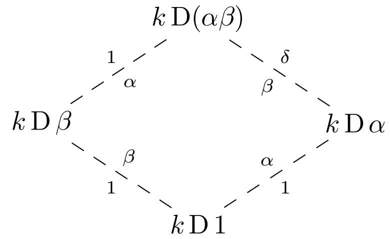
with relations $\rho^{\mathrm{op}} = \{x^{\mathrm{op}}y^{\mathrm{op}} - \delta y^{\mathrm{op}}x^{\mathrm{op}}, (x^{\mathrm{op}})^2, (y^{\mathrm{op}})^2\}$. $A_A, \mathrm{D}\,A_A$ are the following

A_A :



$$\begin{bmatrix} 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&\delta&0 \end{bmatrix} \circlearrowright k^4 \circlearrowleft \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&1&0&0 \end{bmatrix}$$

$\mathrm{D}\,A_A$:



$$\begin{bmatrix} 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&1&0 \end{bmatrix} \circlearrowright k^4 \circlearrowleft \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ \delta&0&0&0\\ 0&1&0&0 \end{bmatrix}$$

(Here, $--$ means the right action). Then every isomorphism from A_A to $\mathrm{D}\,A_A$ is the form $\begin{bmatrix} a&0&0&0\\ b&a&0&0\\ c&0&\delta a&0\\ d&c&b&a \end{bmatrix}$ with $a \in k^\times$. If $\delta = 1$, then $A \cong \mathrm{D}\,A$ as A-bimodules and A is a symmetric k-algebra. Otherwise, $A \ncong \mathrm{D}\,A$ as A-bimodules and A is not a symmetric k-algebra. For n, let $M_n = A(\alpha + (-\delta)^n\beta)$

$$\begin{bmatrix} 0&0\\ (-\delta)^n&0 \end{bmatrix} \circlearrowright k^2 \circlearrowleft \begin{bmatrix} 0&0\\ \delta&0 \end{bmatrix}$$

Then we have an exact sequence

$$\begin{array}{ccccccccc} O & \longrightarrow & A(\alpha + (-\delta)^{n-1}\beta) & \longrightarrow & A & \longrightarrow & A(\alpha + (-\delta)^n\beta) & \longrightarrow & O \\ O & \longrightarrow & M_{n-1} & \longrightarrow & A & \longrightarrow & M_n & \longrightarrow & O \end{array}$$

for each $n \in \mathbb{Z}$, and

$$\begin{aligned} \mathrm{Hom}_A(M_n, A) &= \left\{ \begin{bmatrix} 0&0\\ a&0\\ a(-\delta)^n&0\\ b&a \end{bmatrix} | a, b \in k \right\} \\ \mathrm{Hom}_A(M_m, M_n) &= \{ \begin{bmatrix} a&0\\ b&a \end{bmatrix} | (-\delta)^m a = (-\delta)^n a, a, b \in k \} \end{aligned} \tag{5.3}$$

And we have an exact sequence

$$\begin{array}{ccccccc} O & \longrightarrow & \mathrm{Hom}_A(M_0, M_i) & \longrightarrow & \mathrm{Hom}_A(M_0, A) & \longrightarrow & \\ \mathrm{Hom}_A(M_0, M_{i+1}) & \longrightarrow & \mathrm{Ext}^1_A(M_0, M_i) & \longrightarrow & O & & \end{array}$$

for $i \geq 1$. If $-\delta$ is not a root of 1, then by the equation 5.3 we have

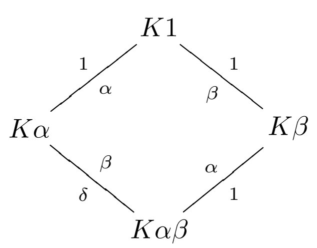
$$\begin{aligned} \dim_k \mathrm{Ext}^1_A(M_0, M_0) &= \dim_k \mathrm{Hom}_A(M_0, M_1) - \dim_k \mathrm{Hom}_A(M_0, A) + \\ &\quad \dim_k \mathrm{Hom}_A(M_0, M_0) \\ &= 1 - 2 + 2 = 1 \\ \dim_k \mathrm{Ext}^i_A(M_0, M_0) &= \dim_k \mathrm{Ext}^1_A(M_0, M_{i-1}) \\ &= \dim_k \mathrm{Hom}_A(M_0, M_{i+1}) - \dim_k \mathrm{Hom}_A(M_0, A) + \\ &\quad \dim_k \mathrm{Hom}_A(M_0, M_{i-1}) \\ &= 1 - 2 + 1 = 0 \\ &\quad \text{for } i \geq 2 \end{aligned} \tag{5.4}$$

Proposition 5.5. *Assume that $-\delta$ is not a root of 1. Let $M = A(\alpha+\beta)$, then we have $\operatorname{Ext}^i_A(M, M) = O$ for all $i \geq 2$.*

Proposition 5.6. *Assume that $-\delta$ is not a root of 1. Let $M = A(\alpha+\beta)$, and $\cdots \to A \to A \to M \to O$ a minimal projective resolution, then all syzygy A-modules $\Omega^n M$ have k-dimension 2, and they are non-isomorphic each other.*

5.2. **The Case of Rings.** Let $A = k(\vec{Q}, \rho)$ be a finite dimensional k-algebra given in §5.1. Let K be a skew field which is a k-algebra, and $B = A^K$. Then $\operatorname{Hom}_K({}_K-, {}_K K)$ and $\operatorname{Hom}_K(-_K, K_K)$ induce a duality between $\operatorname{rep}_{K/k}(\vec{Q}, \rho)$ and $\operatorname{rep}_{K/k}(\vec{Q}^{\mathrm{op}}, \rho^{\mathrm{op}})$. Hence B is a local self-injective artinian ring. According to Theorem 4.2, $\mathsf{Mod}\, B$ is is equivalent to $\mathsf{Rep}_{R/k}(\vec{Q}, \rho)$. For a representation $M = (M, \psi^M)$, $\psi^M(\alpha)$ is a left K-linear map for any arrow α. Then ψ^M is represented by the set of the right multiplications of matrices of K, and their matrix compositions are the opposite compositions of maps (i.e. we take row vectors as elements of K-vector spaces in this subsection). Therefore by taking the transpose of matrices in ${}_A A$ of §5.1, we have a representation ${}_B B$ in $\mathsf{Rep}_{R/k}(\vec{Q}, \rho)$

${}_B B :$



$$\begin{bmatrix} 0&1&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0 \end{bmatrix} \circlearrowright K^4 \circlearrowleft \begin{bmatrix} 0&0&1&0\\ 0&0&0&\delta\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}$$

For $\lambda \in K^\times$, let $M_\lambda = B(\alpha + \lambda\beta)$, then M is represented by

$$\left[\begin{smallmatrix} 0 & \lambda \\ 0 & 0 \end{smallmatrix}\right] \circlearrowright K^2 \circlearrowleft \left[\begin{smallmatrix} 0 & \delta \\ 0 & 0 \end{smallmatrix}\right]$$

Lemma 5.7. *The following hold.*

1. $\operatorname{Hom}_B(M_\lambda, M_\mu) = \{\left[\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right] \mid \lambda a = a\mu, a, b \in K\}$
2. $\operatorname{Hom}_B(M_\lambda, B) = \{\left[\begin{smallmatrix} 0 & a & \lambda a & b \\ 0 & 0 & 0 & a \end{smallmatrix}\right] \mid a, b \in K\}$

Lemma 5.8. *For $n \in \mathbb{Z}$, $\lambda \in K^\times$ and $\delta \in k^\times$, we have an exact sequence*

$$O \to M_{\lambda(-\delta)^n} \xrightarrow{\eta_n} B \xrightarrow{\theta_{n+1}} M_{\lambda(-\delta)^{n+1}} \to O$$

where $\eta_n = \left[\begin{smallmatrix} 0 & 1 & \lambda(-\delta)^n & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right]$, *and* $\theta_{n+1} = \begin{bmatrix} 1 & 0 \\ 0 & \lambda(-\delta)^{n+1} \\ 0 & \delta \\ 0 & 0 \end{bmatrix}$.

Proposition 5.9. *If $\delta \in k^\times$ and $\lambda \in K^\times$ satisfy*

(i) *λ and $\lambda(-\delta)^n$ are not conjugate in $K^\times$ for $n \geq 1$,*
(ii) *for any $n \geq 0$ and any $b \in K$, there exists $a \in K$ such that $\lambda a - a\lambda(-\delta)^n = b$,*

then $\operatorname{Ext}^i_B(M_\lambda, M_\lambda) = 0$ for any $i \geq 1$, and $\operatorname{End}_B(M_\lambda)$ is neither left artinian nor right artinian.

Proof. By Lemma 5.8, for $n \geq 0$, we have an exact sequence

$$O \to M_{\lambda(-\delta)^n} \xrightarrow{\eta_n} B \xrightarrow{\theta_{n+1}} M_{\lambda(-\delta)^{n+1}} \to O.$$

Then in order to prove the first part, it suffices to show that

$$O \to \operatorname{Hom}_B(M_\lambda, M_{\lambda(-\delta)^n}) \xrightarrow{\operatorname{Hom}_B(M_\lambda,\eta_n)} \operatorname{Hom}_B(M_\lambda, B) \xrightarrow{\operatorname{Hom}_B(M_\lambda,\theta_{n+1})} \operatorname{Hom}_B(M_\lambda, M_{\lambda(-\delta)^{n+1}}) \to O.$$

is an exact sequence for $n \geq 0$. By Lemma 5.7 1 and assumption 1 , we have

$$\begin{aligned}\operatorname{Hom}_B(M_\lambda, M_{\lambda(-\delta)^{n+1}}) &= \left\{\left[\begin{smallmatrix} a & b \\ 0 & a\end{smallmatrix}\right] \middle| \lambda a = a\lambda(-\delta)^{n+1}, a, b \in K\right\} \\ &= \left\{\left[\begin{smallmatrix} 0 & b \\ 0 & 0\end{smallmatrix}\right] \middle| b \in K\right\}\end{aligned}$$

According to Lemma 5.7 2, we have

$$\operatorname{Im}\operatorname{Hom}_B(M_\lambda, \theta_{n+1}) = \left\{\left[\begin{smallmatrix} 0 & \lambda a\delta + a\lambda(-\delta)^{n+1} \\ 0 & 0\end{smallmatrix}\right] \middle| a \in K\right\}$$

By assumption 2, there exists $a \in K$ such that $\lambda a - a\lambda(-\delta)^n = b\delta^{-1}$. For the second part, by Lemma 5.7 1, we have

$$\operatorname{End}_B(M_\lambda) = \left\{\left[\begin{smallmatrix} a & b \\ 0 & a\end{smallmatrix}\right] \middle| \lambda a = a\lambda, a, b \in K\right\}$$

Let $\partial_\lambda : K \to K$ be a map defined by $\partial_\lambda(a) = \lambda a - a\lambda$ for $a \in K$. Then ∂_λ is an additive group homomorphism and $F = \operatorname{Ker}\partial_\lambda$ is a skew subfield. For any $s \in F$, $a \in K$, we have

$$\begin{aligned}\partial_\lambda(sa) &= \lambda sa - sa\lambda \\ &= s\lambda a - sa\lambda \\ &= s\partial_\lambda(a)\end{aligned}$$

Therefore K is a left F-vector space and ∂_λ is a left F-linear map. Similarly K is a right F-vector space and ∂_λ is a right F-linear map. We have $\dim_F K = \dim K_F = \infty$, because $O \to F \to K \xrightarrow{\partial_\lambda} K \to O$ is exact. It is easy to see $\operatorname{End}_B(M_\lambda) \cong F \ltimes K$ (this is a trivial extension of F by K). □

Proposition 5.10. *There are a skew field K, its commutative subfield k, $\lambda \in K^\times$ and $\delta \in k^\times$ such that K is a k-algebra and that they satisfy the conditions (i) and (ii) of Proposition 5.9.*

Proof. According to [Co1] or [Co2] Section 8, there are a skew field L and $\lambda \in L$ such that the inner derivation $\partial_\lambda : L \to L$ is surjective. Let K be the skew field $L\{X\}$ of formal Laurant polynomials, and $\delta = -X$. For $0 \neq f = \sum_i \nu_i X^i \in K$, we denote by $\deg_{min} f = \min\{i | \nu_i \neq 0\}$. Then $\deg_{min} f^{-1} = -\deg_{min} f$. Therefore λ and λX^n are not conjugate for $n \geq 1$, because $\deg_{min}\lambda \neq \deg_{min}\lambda X^n$. Let $\partial_{\lambda,n} : K \to K$ be the map defined by $\partial_{\lambda,n}(a) = \lambda a - a\lambda X^n$. Let $g = \sum_i \nu_i X^i \in K$. In the case $n = 0$, there is $\mu_i \in L$ such that $\lambda\mu_i - \mu_i\lambda = \nu_i$. Let $f = \sum_i \mu_i X^i$, then $\partial_{\lambda,0}(f) = g$. In the case $n \geq 1$, $f = \sum_{i=1}^{\infty} \lambda^{-i} g\lambda^{i-1} X^{n(i-1)}$. Hence we have

$$\begin{aligned}\lambda f - f\lambda X^n &= \sum_{i=1}^{\infty} \lambda^{-i+1} g\lambda^{i-1} X^{n(i-1)} - \sum_{i=1}^{\infty} \lambda^{-i} g\lambda^{i} X^{ni} \\ &= g.\end{aligned}$$

We take $k =$ the center $\mathrm{Z}(K)$ of K. Then k satisfies the desired property, because of $X \in \mathrm{Z}(K)$. □

6. Appendix

In this section, we recall some properties of homological algebra without proofs. The reader see e.g. [Ro] for details.

Definition 6.1 (Category)**.** We define a *category* $\mathcal{C}$ by the following data:

1. A class $\mathrm{Ob}\,\mathcal{C}$ of elements called objects of $\mathcal{C}$.
2. For a ordered pair (X, Y) of objects a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms is given such that $\mathrm{Hom}_{\mathcal{C}}(X, Y) \cap \mathrm{Hom}_{\mathcal{C}}(X', Y') = \phi$ for $(X, Y) \neq (X', Y')$ (an element f of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is called a morphism, and denote by $f : X \to Y$).
3. For each triple (X, Y, Z) of objects of $\mathcal{C}$ a map
$$\theta(X, Y, Z) : \mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$
(θ is called the composition map) is given.
4. The composition map θ is associative.
5. For each object X of $\mathcal{C}$, there is a morphism $1_X : X \to X$ such that for any $g : Y \to X$, $h : X \to Z$ we have $1_X g = g, h1_X = h$.

Definition 6.2 (Complex)**.** A diagram $X^{\boldsymbol{\cdot}} : \ldots \to X^{i-1} \xrightarrow{d^{i-1}} X^i \xrightarrow{d^i} X^{i+1} \to \ldots$ is called a (*cochain*) *complex* if $d^{i+1}d^i = 0$ for all i, that is, $\mathrm{Im}\, d^{i-1} \subset \mathrm{Ker}\, d^i$ for all i. A complex $X^{\boldsymbol{\cdot}}$ is called *exact* if $\mathrm{Im}\, d^{i-1} = \mathrm{Ker}\, d^i$ for all i. Sometimes, we call an exact sequence for an exact complex. For a complex $X^{\boldsymbol{\cdot}}$, $\mathrm{H}^n(X^{\boldsymbol{\cdot}}) = \mathrm{Ker}\, d^n / \mathrm{Im}\, d^n$ is called the n-th *cohomology*.

Lemma 6.3. *Let $O \to V_0 \to V_1 \to \ldots \to V_n \to O$ be an exact sequence of k-vector spaces. Then we have*
$$\dim_k V_0 = \Sigma_{i=1}^n (-1)^i \dim_k V_i.$$

Definition 6.4. For $f : X \to Y$ in $\mathsf{Mod}\, A$, $\mathrm{Hom}_A(X, Y) =$ the set of left A-linear maps from X to Y. For $M \in \mathsf{Mod}\, A$, we have the following additive group homomorphisms
$$\mathrm{Hom}_A(M, X) \xrightarrow{\mathrm{Hom}_A(M,f)} \mathrm{Hom}_A(M, Y)(g \mapsto f \circ g)$$
$$\mathrm{Hom}_A(Y, M) \xrightarrow{\mathrm{Hom}_A(M,f)} \mathrm{Hom}_A(X, M)(h \mapsto h \circ f).$$

Definition 6.5 (Projective, Injective Module)**.** A left A-module M is called A-*projective* if for any surjective A-linear map $X \to Y$ we have a surjective additive group homomorphism $\mathrm{Hom}_A(M, X) \xrightarrow{\mathrm{Hom}_A(M,f)} \mathrm{Hom}_A(M, Y)$. Similarly, a left A-module M is called A-*injective* if for any injective A-linear map $X \to Y$ we have a surjective additive group homomorphism $\mathrm{Hom}_A(Y, M) \xrightarrow{\mathrm{Hom}_A(M,f)} \mathrm{Hom}_A(X, M)$.

Proposition 6.6. *A left A-module A is A-projective. In the case of A being a finite dimensional k-algebra, $\mathrm{D}\, A$ is a injective left A-module.*

Proposition 6.7. *For a left A-module M, the following hold.*

1. *M is A-projective if and only if any surjective A-linear map $f : X \to M$ splits (i.e. there exists $g : M \to X$ such that $gf = 1_M$).*
2. *M is A-injective if and only if any injective A-linear map $f : M \to Y$ splits (i.e. there exists $g : Y \to M$ such that $fg = 1_M$).*

Proposition 6.8. *For a left A-module M, the following hold.*

1. *There exists a set I and $f : A^{(I)} \to M$ such that f is surjective.*

2. *There exists a injective A-module E and $g : M \to E$ such that g is injective.*

Definition 6.9 (Projective, Injective Resolution)**.** For a left A-module M, according to Proposition 6.8, we have a surjective A-linear map $\epsilon_0 : P_0 \to M$ with P_0 being A-projective. For $\operatorname{Ker}\epsilon_0$, we have a surjective A-linear map $\epsilon_1 : P_1 \to \operatorname{Ker}\epsilon_0$ with P_1 being A-projective. Therefore we have an exact complex

$$\ldots \to P_n \to \ldots \to P_1 \to P_0 \to M \to O,$$

with P_i being A-projective The complex $P_\bullet : \ldots \to P_n \to \ldots \to P_1 \to P_0$ is called *projective resolution* of M.

Similarly, we have an exact complex

$$O \to M \to I^0 \to I^1 \to \ldots \to I^n \to \ldots ,$$

with I^i being A-injective The complex $I^\bullet : I^0 \to I^1 \to \ldots \to I^n \to \ldots$ is called *injective resolution* of M.

When we have a projective resolution

$$O \to P_n \to \ldots \to P_1 \to P_0 \to M \to O,$$

we say that the *projective dimension* of M is at most n, denote by $\operatorname{pdim}_A M \leq n$. Similarly, when we have an injective resolution

$$O \to M \to I^0 \to I^1 \to \ldots \to I^n \to O,$$

we say that the *injective dimension* of M is at most n, denote by $\operatorname{idim}_A M \leq n$.

The *left global dimension* $\operatorname{lgldim} A$ of A is the supremum of $\operatorname{pdim} M$ of left A-modules M.

Theorem 6.10 (Higher Extension Groups)**.** *The following hold.*

1. *Let $\ldots \to P_n \to \ldots \to P_1 \to P_0 \to X \to O$ be a projective resolution of a left A-module X. Then for any $Y \in \mathsf{Mod}\, A$ and any $n \geq 0$, $\mathrm{H}^n \operatorname{Hom}_A(P_\bullet, Y)$ is determined independent of choice of projective resolutions.*
2. *Let $O \to Y \to I^0 \to I^1 \to \ldots \to I^n \to \ldots$ be an injective resolution of a left A-module Y. Then for any $M \in \mathsf{Mod}\, A$ and any $n \geq 0$, $\mathrm{H}^n \operatorname{Hom}_A(X, I^\bullet)$ is determined independent of choice of injective resolutions.*
3. *For $X, Y \in \mathsf{Mod}\, A$, we have $\mathrm{H}^n \operatorname{Hom}_A(P_{X\bullet}, Y) \cong \mathrm{H}^n \operatorname{Hom}_A(X, I_Y^\bullet)$ for $n \geq 0$, where $P_{X\bullet}$ (resp., $I_Y^\bullet$) is a projective (resp., an injective) resolution of X (resp., Y).*

The additive group $\mathrm{H}^n \operatorname{Hom}_A(P_{X\bullet}, Y) \cong \mathrm{H}^n \operatorname{Hom}_A(X, I_Y^\bullet)$ is called the n-th Extension group $\operatorname{Ext}_A^n(X, Y)$.

Proposition 6.11. *The following hold.*

1. *If P is A-projective, then $\operatorname{Ext}_A^n(P, Y) = 0$ for $n \geq 1$.*
2. *If I is A-injective, then $\operatorname{Ext}_A^n(X, I) = 0$ for $n \geq 1$.*
3. *For an exact sequence $O \to X \to Y \to Z \to O$ in $\mathsf{Mod}\, A$, we have long exact sequences*

$$\begin{aligned} O \to \operatorname{Hom}_A(M, X) \to \quad & \operatorname{Hom}_A(M, Y) \to \operatorname{Hom}_A(M, Z) \to \\ \operatorname{Ext}_A^1(M, X) \to \quad & \operatorname{Ext}_A^1(M, X) \to \operatorname{Ext}_A^1(M, X) \to \\ \operatorname{Ext}_A^2(M, X) \to \ldots , & \end{aligned}$$

and

$$\begin{aligned} O \to \operatorname{Hom}_A(Z, M) \to \quad & \operatorname{Hom}_A(Y, M) \to \operatorname{Hom}_A(X, M) \to \\ \operatorname{Ext}^1_A(Z, M) \to \quad & \operatorname{Ext}^1_A(Y, M) \to \operatorname{Ext}^1_A(X, M) \to \\ \operatorname{Ext}^2_A(Z, M) \to \dots . & \end{aligned}$$

Lemma 6.12 (Nakayama's Lemma)**.** *Let A be a ring with unity, J the Jacobson radical of A, and M a finitely generated left A-module. For a left A-submodule N of M, if $JM + N = M$, then $N = M$.*

Definition 6.13 (Minimal Projective resolution)**.** Let M be a finitely generated left A-module. A projective resolution of M

$$\dots \to P_n \to \dots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to M \to O$$

is called a *minimal projective resolution* provided that $\operatorname{Im} d_i \subset JP_{i-1}$ for all $i \geq 1$. This resolution does not exists in general. In the case of A being left artinian, a minimal projective resolution exists for any finitely generated left A-module.

Definition 6.14 (Indecomposable Module)**.** A left A-module M is called *indecomposable* provided that if $M = X \oplus Y$, then X or $Y = O$.

Definition 6.15. Let A and B be k-algebras. The tensor product $A \otimes_k B$ is the k-algebra defined by

$$\begin{aligned} (a \otimes b)(a' \otimes b') &= aa' \otimes bb' \\ 1_{A \otimes B} &= 1_A \otimes 1_B. \end{aligned}$$

Then we have

$$\begin{aligned} (1_A \otimes b)(a \otimes 1_B) &= a \otimes b \\ &= (a \otimes 1_B)(1_A \otimes b). \end{aligned}$$

Definition 6.16 (The Skew Field of Formal Laurant Polynomials)**.** For a skew field L, let

$$L\{X\} = \{\Sigma_{i=n}^{\infty} a_i X^i | n \in \mathbb{Z}, a_i \in L\}.$$

We define the multiplication of $\Sigma_{i=m}^{\infty} a_i X^i, \Sigma_{j=n}^{\infty} b_j X^j \in L\{X\}$ by

$$(\Sigma_{i=m}^{\infty} a_i X^i)(\Sigma_{j=n}^{\infty} b_j X^j) = \Sigma_{k=m+n}^{\infty} (\Sigma_{i+j=k} a_i b_j) X^k,$$

and define

$$\deg_{min}(\Sigma_{i=m}^{\infty} a_i X^i) = m$$

if $a_m \neq 0$. Then we have

$$\deg_{min}(fg) = \deg_{min}(f) + \deg_{min}(g)$$

for non-zero polynomials $f, g \in L\{X\}$. It is easy to see that $L\{X\}$ is a skew field.

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