Diagonalization of three-dimensional pseudo-Riemannian metrics

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Abstract

We give a short proof of the fact that any Riemannian or Lorentzian real analytic metric in dimension 3 can be locally adapted to the diagonal form. We use the classical Cauchy-Kowalevski Theorem to this purpose.

Introduction

The problem of local diagonalization of a Riemannian metric in the three-dimensional real analytic case is classical (see *e.g.* Eisenhart [4] or, for a more modern treatment, Bryant *et al.* [1]. It has been shown (DeTurck and Yang [2]) that a C^∞-Riemannian 3-metric always admits diagonalization. An essential contribution in the higher-dimensional case is that by Tod [8]. Here Riemannian or Lorentzian metric with $n > 3$ is considered. It is shown that diagonalizability of the metric generically imposes restrictions on the third derivative of the Weyl tensor when $n = 4$, the first derivative of the Weyl tensor when $n = 5$ and the Weyl tensor itself when $n > 5$. It is also shown that some of the plane-wave metrics provide examples of four-dimensional non-diagonalizable Lorentzian metrics. The following classical theorem is well-known:

Theorem A. *Let (M, g) be a real analytic three-dimensional Riemannian manifold. Then, in a neighborhood of each point $p \in M$, there is a system (x, y, z) of local coordinates in which g adopts a diagonal form. All coordinate transformations for which the diagonality of g is preserved depend on three arbitrary real analytic functions of two variables.*

As communicated to us privately by P. Tod (and, thanks to him, also by J. Grant), a modification of this theorem to the Lorentzian case should be rather easy. Yet, the authors mentioned above do not remember seeing any published version of the proof. The aim of our paper is to give a short proof of such a modification in the classical context.

Main Theorem. *Let (M, g) be a real analytic three-dimensional pseudo-Riemannian manifold. Then, in a neighborhood of each point $p \in M$, there is a system (x, y, z) of local coordinates in*

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which g adopts a diagonal form. All coordinate transformations for which the diagonality of g is preserved depend (locally) on three arbitrary real analytic functions of two variables.

The proof is the aim of the next Section. Yet, we shall recall here the classical Cauchy-Kowalevski Theorem. A modern presentation of this theorem can be found *e.g.* in [3, 7]. Yet, here we shall present its original version.

Theorem B ([5]). *Let*

$$(0.1)\quad \frac{\partial^{r_i} u_i}{\partial {x_1}^{r_i}} = H_i(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m, q), \quad i = 1, 2, \dots, m,$$

be a system of partial differential equations for unknown functions $u_i = u_i(x_1, x_2, \dots, x_n)$, $i = 1, 2, \dots, m$, of independent variables x_1, x_2, …, x_n defined on an open set of the number space $\mathbb{R}^n$, where q denotes the set of partial derivatives

$$\Big(\frac{\partial^{i_1+i_2+\cdots+i_n} u_j}{\partial {x_1}^{i_1} \partial {x_2}^{i_2} \cdots \partial {x_n}^{i_n}}\ ;\ j = 1, 2, \dots, m,\ 1 \le i_1 + i_2 + \cdots + i_n \le r_j,\ i_1 \ne r_j\Big)$$

and H_i, $i = 1, 2, \dots, m$, are real analytic at

$$\begin{aligned}
&x_\alpha = a_\alpha,\ \alpha = 1, 2, \dots, n,\ u_j = b_j,\ j = 1, 2, \dots, m,\\
&\frac{\partial^{i_1+i_2+\cdots+i_n} u_j}{\partial {x_1}^{i_1} \partial {x_2}^{i_2} \cdots \partial {x_n}^{i_n}} = c_{j,i_1,i_2,\dots,i_n},\\
&j = 1, 2, \dots, m,\ 1 \le i_1 + i_2 + \cdots + i_n \le r_j,\ i_1 \ne r_j.
\end{aligned}$$

Moreover let functions $\varphi_{i\lambda} = \varphi_{i\lambda}(x_2, x_3, \dots, x_n)$, $i = 1, 2, \dots, m$, $\lambda = 0, 1, \dots, r_i - 1$, be real analytic at $(a_1, a_2, \dots, a_n)$ and satisfy

$$\begin{aligned}
&\frac{\partial^{i_1+i_2+\cdots+i_n} \varphi_{i\lambda}}{\partial {x_2}^{i_2} \partial {x_3}^{i_3} \cdots \partial {x_n}^{i_n}}(a_1, a_2, \dots, a_n) = c_{i,\lambda,i_2,\dots,i_n},\\
&i = 1, 2, \dots, m,\ \lambda = 0, 1, \dots, r_i - 1,\ 1 \le i_2 + i_3 + \cdots + i_n \le r_i - \lambda,\\
&c_{i,0,0,\dots,0} = b_i,\ i = 1, 2, \dots, m.
\end{aligned}$$

Then the system (0.1) *has unique solutions*

$$u_i = \Phi_i(x_1, x_2, \dots, x_n), \quad i = 1, 2, \dots, m,$$

where Φ_i, $i = 1, 2, \dots, m$, are real analytic at $(a_1, a_2, \dots, a_n)$, and satisfy

$$\begin{aligned}
&\frac{\partial^\lambda \Phi_i}{\partial {x_1}^\lambda}(a_1, x_2, \dots, x_n) = \varphi_{i\lambda}(x_2, x_3, \dots, x_n),\\
&i = 1, 2, \dots, m,\ \lambda = 0, 1, \dots, r_i - 1.
\end{aligned}$$

1 Proof of the Main Theorem

Let M be a three-dimensional real analytic manifold, and g a pseudo-Riemannian metric on M. Since our problem is only local, we can assume that (M, g) is of the form $(\mathcal{U}, g)$, where $\mathcal{U} \subset \mathbb{R}^3[x^1, x^2 x^3]$ is an open domain and g is a pseudo-Riemannian metric on $\mathcal{U}$, *i.e.,* of the form

$$(1.1)\quad g = \sum_{i,j=1}^{3} g_{ij} \mathrm{d}x^i \mathrm{d}x^j,$$

where $g_{ij} = g_{ij}(x^1, x^2, x^3)$, $i, j = 1, 2, 3$, are entries of a symmetric regular matrix. We shall use the Cauchy-Kowalevski Theorem to show that, in a neighborhood of any fixed point $p \in \mathcal{U}$, there exists a local coordinate system (u, v, w) diagonalizing the expression of g. Then we can write the Cartesian coordinate functions x^1, x^2 and x^3 in the form

$$x^1 = F(u, v, w), \; x^2 = G(u, v, w), \; x^3 = H(u, v, w), \tag{1.2}$$

where F, G and H are real analytic functions of u, v and w. Now we have

$$\begin{aligned} \mathrm{d}x^1 &= F_u \mathrm{d}u + F_v \mathrm{d}v + F_w \mathrm{d}w, \\ \mathrm{d}x^2 &= G_u \mathrm{d}u + G_v \mathrm{d}v + G_w \mathrm{d}w, \\ \mathrm{d}x^3 &= H_u \mathrm{d}u + H_v \mathrm{d}v + H_w \mathrm{d}w, \end{aligned} \tag{1.3}$$

where the lower indices u, v and w of F, G and H indicate the first partial derivatives of the corresponding functions.

Substituting (1.3) into (1.1), we obtain the expression for g in the new coordinates u, v and w, let us say

$$\begin{aligned} g = {} & G_{11}(\mathrm{d}u)^2 + 2G_{12}\mathrm{d}u\mathrm{d}v + 2G_{13}\mathrm{d}u\mathrm{d}w \\ & + G_{22}(\mathrm{d}v)^2 + 2G_{23}\mathrm{d}v\mathrm{d}w + G_{33}(\mathrm{d}w)^2 \end{aligned} \tag{1.4}$$

and we shall express explicitly the diagonality conditions $G_{12} = G_{13} = G_{23} = 0$. We get the following three partial differential equations of first order which are quadratic and homogeneous with respect to the first derivatives of the unknown functions F, G and H:

$$\begin{aligned} & g_{11}F_uF_v + g_{12}(F_uG_v + F_vG_u) + g_{13}(F_uH_v + F_vH_u) \\ & \quad + g_{22}G_uG_v + g_{23}(G_uH_v + G_vH_u) + g_{33}H_uH_v = 0, \end{aligned} \tag{1.5}$$

$$\begin{aligned} & g_{11}F_uF_w + g_{12}(F_uG_w + F_wG_u) + g_{13}(F_uH_w + F_wH_u) \\ & \quad + g_{22}G_uG_w + g_{23}(G_uH_w + G_wH_u) + g_{33}H_uH_w = 0, \end{aligned} \tag{1.6}$$

$$\begin{aligned} & g_{11}F_vF_w + g_{12}(F_vG_w + F_wG_v) + g_{13}(F_vH_w + F_wH_v) \\ & \quad + g_{22}G_vG_w + g_{23}(G_vH_w + G_wH_v) + g_{33}H_vH_w = 0. \end{aligned} \tag{1.7}$$

This system of partial differential equations is obviously not of the form to which the Cauchy-Kowalevski Theorem can be applied. Namely, according to (0.1), we need to express all three first derivatives of the functions F, G and H with respect to one of the variables u, v or w, which is not possible here. But we can help ourselves in this situation using a simple transformation of independent variables. (See [6] for an analogous situation). We define new functions U, V and W of the variables $x = w$, $y = v$ and $z = u + v + w$ by

$$U(x, y, z) = F(u, v, w), \; V(x, y, z) = G(u, v, w), \; W(x, y, z) = H(u, v, w).$$

Hence we have

$$
(1.8)\quad
\begin{aligned}
F_u(u,v,w) &= U_z(w,v,u+v+w),\\
F_v(u,v,w) &= U_y(w,v,u+v+w) + U_z(w,v,u+v+w),\\
F_w(u,v,w) &= U_x(w,v,u+v+w) + U_z(w,v,u+v+w),\\
G_u(u,v,w) &= V_z(w,v,u+v+w),\\
G_v(u,v,w) &= V_y(w,v,u+v+w) + V_z(w,v,u+v+w),\\
G_w(u,v,w) &= V_x(w,v,u+v+w) + V_z(w,v,u+v+w),\\
H_u(u,v,w) &= W_z(w,v,u+v+w),\\
H_v(u,v,w) &= W_y(w,v,u+v+w) + W_z(w,v,u+v+w),\\
H_w(u,v,w) &= W_x(w,v,u+v+w) + W_z(w,v,u+v+w).
\end{aligned}
$$

Substituting these expressions into (1.5)–(1.7), we obtain a new system of PDE's for our problem evaluated at $(x,y,z) = (w,v,u+v+w)$. Here we shall use a brief notation for the partial derivatives of U, V and W, and also for the new metric components $\bar{g}_{ij}(x,y,z) = g_{ij}(z-y-x,y,x)$, $i,j = 1,2,3$, defined in the same domain as U, V and W. The new equations will be again quadratic with respect to the first derivatives but not more homogeneous. Indeed, we get three PDE's

$$
(1.9)\quad
\begin{aligned}
&\bar{g}_{11}(U_z)^2 + 2\bar{g}_{12}U_zV_z + 2\bar{g}_{13}U_zW_z + \bar{g}_{22}(V_z)^2 + 2\bar{g}_{23}V_zW_z + \bar{g}_{33}(W_z)^2\\
&\qquad + A(y)U_z + B(y)V_z + C(y)W_z = 0,
\end{aligned}
$$

$$
(1.10)\quad
\begin{aligned}
&\bar{g}_{11}(U_z)^2 + 2\bar{g}_{12}U_zV_z + 2\bar{g}_{13}U_zW_z + \bar{g}_{22}(V_z)^2 + 2\bar{g}_{23}V_zW_z + \bar{g}_{33}(W_z)^2\\
&\qquad + A(x)U_z + B(x)V_z + C(x)W_z = 0,
\end{aligned}
$$

$$
(1.11)\quad
\begin{aligned}
&\bar{g}_{11}(U_z)^2 + 2\bar{g}_{12}U_zV_z + 2\bar{g}_{13}U_zW_z + \bar{g}_{22}(V_z)^2 + 2\bar{g}_{23}V_zW_z + \bar{g}_{33}(W_z)^2\\
&\qquad + A(x)U_z + B(x)V_z + C(x)W_z + A(y)U_z + B(y)V_z + C(y)W_z + D = 0,
\end{aligned}
$$

where we put

$$
(1.12)\quad
\begin{aligned}
A(x) &= \bar{g}_{11}U_x + \bar{g}_{12}V_x + \bar{g}_{13}W_x,\\
B(x) &= \bar{g}_{21}U_x + \bar{g}_{22}V_x + \bar{g}_{23}W_x,\\
C(x) &= \bar{g}_{31}U_x + \bar{g}_{32}V_x + \bar{g}_{33}W_x,\\
A(y) &= \bar{g}_{11}U_y + \bar{g}_{12}V_y + \bar{g}_{13}W_y,\\
B(y) &= \bar{g}_{21}U_y + \bar{g}_{22}V_y + \bar{g}_{23}W_y,\\
C(y) &= \bar{g}_{31}U_y + \bar{g}_{32}V_y + \bar{g}_{33}W_y,
\end{aligned}
$$

and

$$
D = D(x,y) = A(y)U_x + B(y)V_x + C(y)W_x.
$$

Subtracting (1.9) from (1.10) we obtain

$$
(1.13)\quad \{A(x) - A(y)\}U_z + \{B(x) - B(y)\}V_z + \{C(x) - C(y)\}W_z = 0.
$$

Also, subtracting (1.10) from (1.11), we obtain

$$
(1.14)\quad A(y)U_z + B(y)V_z + C(y)W_z + D = 0,
$$

and hence

(1.15) $W_z = -(A(y)U_z + B(y)V_z + D)/C(y)$.

We can assume here $C(y) \neq 0$ as we shall see later. Substituting (1.15) into (1.9) and (1.13), we obtain

$$\begin{aligned} &\{\bar{g}_{11}C(y)^2 - 2\bar{g}_{13}A(y)C(y) + \bar{g}_{33}A(y)^2\}(U_z)^2 \\ &\quad + 2\{\bar{g}_{12}C(y)^2 - \bar{g}_{13}B(y)C(y) - \bar{g}_{23}A(y)C(y) + \bar{g}_{33}A(y)B(y)\}U_zV_z \\ &\quad + \{\bar{g}_{22}C(y)^2 - 2\bar{g}_{23}B(y)C(y) + \bar{g}_{33}B(y)^2\}(V_z)^2 \\ &\quad - 2\{\bar{g}_{13}C(y)D - \bar{g}_{33}A(y)D\}U_z - 2\{\bar{g}_{23}C(y)D - \bar{g}_{33}B(y)D\}V_z \\ &\quad + \bar{g}_{33}D^2 - C(y)^2D = 0 \end{aligned} \tag{1.16}$$

and

$$\begin{aligned} &\{A(x)C(y) - A(y)C(x)\}U_z + \{B(x)C(y) - B(y)C(x)\}V_z \\ &\quad + \{C(y) - C(x)\}D = 0. \end{aligned} \tag{1.17}$$

We now have to solve the system of two PDE's, one quadratic (1.16) and one linear (1.17), with respect to the functions U_z and V_z. If this system can be solved in an explicit form

(1.18) $U_z = T_1,\ V_z = T_2,$

where T_1 and T_2 are real analytic functions of x, y, z, U, V, W and of the derivatives U_x, U_y, V_x, V_y, then, after the substitution into (1.14), we obtain a system of three PDE's expressed with respect to U_z, V_z and W_z in the form where the Cauchy-Kowalevski Theorem can be directly applied.

Recall the Cauchy initial conditions for this case. Let (x_0, y_0, z_0) be a point from the definition domain of the functions U, V and W. Define three functions of two variables x and y (real analytic but arbitrary) in a neighborhood of (x_0, y_0) in the (x, y)-plane by the formulas

(1.19) $M_1(x, y) = U(x, y, z_0),\ M_2(x, y) = V(x, y, z_0),\ M_3(x, y) = W(x, y, z_0).$

Further, denote for a moment x, y and z as x_1, x_2 and x_3. We shall define constants

$$a_i = M_i(x_0, y_0),\ a_{i,j} = \frac{\partial M_i}{\partial x_j}(x_0, y_0), \quad i = 1, 2, 3,\ j = 1, 2.$$

These constants are obviously arbitrary parameters (the Taylor coefficients of the expansions of M_i, $i = 1, 2, 3$, of degree zero and one, respectively). We can rewrite them in the form $a_1 = U(x_0, y_0, z_0)$, $a_2 = V(x_0, y_0, z_0)$, $a_3 = W(x_0, y_0, z_0)$, $a_{1,1} = U_x(x_0, y_0, z_0)$, $a_{2,1} = V_x(x_0, y_0, z_0)$, $a_{3,1} = W_x(x_0, y_0, z_0)$, $a_{1,2} = U_y(x_0, y_0, z_0)$, $a_{2,2} = V_y(x_0, y_0, z_0)$ and $a_{3,2} = W_y(x_0, y_0, z_0)$.

We shall now need the following Lemma that asserts existence of real quadrics on the (U_z, V_z)-plane.

Lemma 1.1. *We can choose our initial conditions at* (x_0, y_0, z_0) *so that the equation* (1.16) *defines a real quadric in the* (U_z, V_z)*-plane.*

Proof of the Lemma. We treat our objects at a fixed point (x_0, y_0, z_0), and we shall use an abbreviate notation everywhere. For example, we denote by $\bar{g}_{ij}$ the values $\bar{g}_{ij}(x_0, y_0, z_0)$, $i, j = 1, 2, 3$,

By (1.12) we have

$$
\begin{aligned}
A(y) &= \bar{g}_{11}a_{1,2} + \bar{g}_{12}a_{2,2} + \bar{g}_{13}a_{3,2},\\
B(y) &= \bar{g}_{21}a_{1,2} + \bar{g}_{22}a_{2,2} + \bar{g}_{23}a_{3,2},\\
C(y) &= \bar{g}_{31}a_{1,2} + \bar{g}_{32}a_{2,2} + \bar{g}_{33}a_{3,2},\\
D &= A(y)a_{1,1} + B(y)a_{2,1} + C(y)a_{3,1}.
\end{aligned}
$$

Due to the regularity of the matrix $(\bar{g}_{ij})$, $i, j = 1, 2, 3$, we can choose the initial conditions $a_{1,2}$, $a_{2,2}$ and $a_{3,2}$ so that $A(y) = 0$, $B(y) = 0$ and $C(y) > 0$. For this choice of initial conditions we have $D = C(y)a_{3,1}$.

Now our quadratic equation (1.16) reduces to

$$(1.20)\quad {}^t\boldsymbol{x}G\boldsymbol{x} - 2\,{}^t\boldsymbol{b}\boldsymbol{x} + c = 0,$$

where

$$\boldsymbol{x} = \begin{pmatrix} U_z \\ V_z \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} \bar{g}_{13}a_{3,1} \\ \bar{g}_{23}a_{3,1} \end{pmatrix}, \quad G = \begin{pmatrix} \bar{g}_{11} & \bar{g}_{12} \\ \bar{g}_{12} & \bar{g}_{22} \end{pmatrix}, \quad c = \bar{g}_{33}(a_{3,1})^2 - D$$

and ${}^t\boldsymbol{x}$ is the transpose of the vector $\boldsymbol{x}$.

Since our metric g is non-degenerate, we see that G is not a null matrix; otherwise we get a contradiction with the regularity of the matrix $(\bar{g}_{ij})$, $i, j = 1, 2, 3$. Hence the rank of G is 2 or 1.

Firstly, if $\mathrm{rank}(G) = 2$, then the eigenvalues λ_1 and λ_2 of G are non-zero. Now we put $\bar{\boldsymbol{x}} = \boldsymbol{x} - G^{-1}\boldsymbol{b}$, where G^{-1} is the inverse matrix of G, and $\tilde{\boldsymbol{x}} = {}^tT\bar{\boldsymbol{x}}$, where T is a fixed orthogonal matrix which diagonalizes G. Then (1.20) reduces to

$$(1.21)\quad {}^t\tilde{\boldsymbol{x}}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\tilde{\boldsymbol{x}} + \frac{1}{\lambda_1\lambda_2}\det\begin{pmatrix} G & -\boldsymbol{b} \\ -\,{}^t\boldsymbol{b} & c \end{pmatrix} = 0.$$

Now, if the eigenvalues λ_1 and λ_2 have opposite signs, our quadric is real. Assume next that λ_1 and λ_2 are both positive. We need to prove that the second term of (1.21) can be made negative. Indeed, the initial Cauchy condition $a_{3,1}$ can be chosen positive and as small as needed. Let us develop the determinant of degree 3 in (1.21) with respect to the first two lines. We obtain the sum of $(1/\lambda_1\lambda_2)(\det G)c = c$ and two other terms which are tending to zero if $a_{3,1}$ is tending to zero. Moreover, c has the same sign as $-D$ for $a_{3,1}$ tending to zero. Since $C(y) > 0$, we get $D > 0$.

Secondly, if $\mathrm{rank}(G) = 1$, then one of eigenvalues of G is non-zero, say $\lambda_1 \neq 0$ and $\lambda_2 = 0$. Now we put $\bar{\boldsymbol{x}} = {}^tT\boldsymbol{x}$ and $\bar{\boldsymbol{b}} = {}^tT\boldsymbol{b}$, where T is a fixed orthogonal matrix which diagonalizes G. Then (1.20) reduces to

$${}^t\bar{\boldsymbol{x}}\begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \end{pmatrix}\bar{\boldsymbol{x}} - 2\,{}^t\bar{\boldsymbol{b}}\bar{\boldsymbol{x}} + c = 0,$$

that is,

$$\lambda_1\bar{x}_1{}^2 - 2(\bar{b}_1\bar{x}_1 + \bar{b}_2\bar{x}_2) + c = 0,$$

where

$$\begin{pmatrix}\bar{x}_1\\ \bar{x}_2\end{pmatrix} = \bar{\boldsymbol{x}}, \quad \begin{pmatrix}\bar{b}_1\\ \bar{b}_2\end{pmatrix} = \bar{\boldsymbol{b}}.$$

We see that the both components $\bar{b}_1$ and $\bar{b}_2$ of $\bar{\boldsymbol{b}}$ are tending to zero when $a_{3,1}$ is tending to zero. Hence we can conclude the argument analogously to the first case. We only remark that the quadric is now either a parabola or a couple of straight lines.

This completes the proof of the Lemma. ∎

Now, if we choose the initial conditions as in the proof of Lemma 1.1, the formula (1.17) simplifies to

$$A(x)C(y)U_z + B(x)C(y)V_z + \{C(y) - C(x)\}D = 0.$$

We see easily that, choosing the initial conditions properly, we can assume that this line is an arbitrary line in the (U_z, V_z)-plane.

Suppose now that the Cauchy initial conditions are chosen in such a way that the corresponding real quadric (1.16) and the corresponding line (1.17) intersect transversally. If we now fix one of the intersection points, let us say $(U_z(x_0, y_0, z_0), V_z(x_0, y_0, z_0))$, then, due to the implicit function theorem for the real analytic case, the derivatives U_z and V_z are real analytic functions of the x, y, z, U, V, W, U_x, U_y, V_x and V_y in a neighborhood of (x_0, y_0, z_0) with the above prescribed values at (x_0, y_0, z_0). Hence (1.18) is satisfied and the Cauchy-Kowalevski Theorem can be applied to the whole system of PDE's. Hence the Main Theorem follows.

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