Pseudo-symmetric spaces of constant type in dimension three

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Contents

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Abstract

Pseudo-symmetric spaces of constant type in dimension three are Riemannian manifolds of dimension three whose Ricci tensor has, at all points, one double eigenvalue and one simple constant eigenvalue. The explicit classification of such spaces with nonzero constant eigenvalue is given.

Introduction

According to [6], a Riemannian manifold (M, g) is said to be *pseudo-symmetric* if the following formula holds for arbitrary vector fields X and Y on M:

(0.1) $R(X, Y) \cdot R = F((X \wedge Y) \cdot R)$,

where

a) R denotes the Riemannian curvature tensor of type (1,3) on (M, g) and

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

denote the corresponding curvature transformations,

b) $X \wedge Y$ denotes the endomorphism of the tangent bundle TM defined by

(0.2) $(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y$,

c) F is a smooth function on M,

d) the dot in each side of the formula (0.1) denotes the derivation on the tensor algebra of TM induced by an endomorphism of this tangent bundle.

We call a pseudo-symmetric space (M, g) *of constant type* if $F = \tilde{c} =$ constant.

Using the Introduction in [15] (where a result of [5] plays an essential role) we obtain easily the following characterization in dimension three:

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Proposition 0.1. *A three-dimensional Riemannian manifold* (M, g) *is pseudo-symmetric of constant type* $F = \tilde{c}$ *if and only if its principal Ricci curvatures* ρ_1, ρ_2 *and* ρ_3 *locally satisfy the following conditions* (*up to a numeration*):

(1) $\rho_3 = 2\tilde{c}$,

(2) $\rho_1 = \rho_2$ *everywhere.*

We are not interested in the case when (M, g) is a space of constant curvature and therefore we assume always $\rho_1 = \rho_2 \neq \rho_3$.

If $\tilde{c} = 0$, and hence $F = 0$, we obtain a definition of *semi-symmetric space*. The theory of semi-symmetric spaces has been developed in [20], [21], [22], [10], [7], [1], [2] and especially in the book [3]. For the three-dimensional case, see the explicit classification in [10], [7] and [3, Chapter 6]. We exclude this case from our considerations.

For $\tilde{c} \neq 0$, the present authors made an explicit classification in [15] for so-called "asymptotically foliated" (or "non-elliptic") spaces in dimension three. They left aside a number of singular cases. (See Section 4 for the terminology). In [16] they treated the more complicated "elliptic" spaces in the full generality. The aim of this paper is two-fold:

a) To complete the classification in [15] by singular cases and to unify the notation used there separately for $\tilde{c} < 0$ and $\tilde{c} > 0$.

b) To join the study of non-elliptic spaces and elliptic spaces in one comprehensive article.

A computer check (the software "*Mathematica*" by Wolfram Research Inc.) was also used during this work.

1 The basic system of partial differential equations for the problem

Let (M, g) be a three-dimensional Riemannian manifold whose Ricci tensor $\hat{R}$ has eigenvalues $\rho_1 = \rho_2 \neq \rho_3$ with nonzero constant ρ_3. Choose a neighborhood $\tilde{U}$ of a fixed point $m \in M$ and a smooth vector field E_3 of unit eigenvectors corresponding to the Ricci eigenvalue ρ_3 in $\tilde{U}$. Let $S : D^2 \to \tilde{U}$ be a surface through m which is transversal with respect to all trajectories generated by E_3 at all cross-points and not orthogonal to such a trajectory at m. (The vector field E_3 determines an orientation of S). Then there is a normal neighborhood U of m, $U \subset \tilde{U}$, with the property that each point $p \in U$ is projected to exactly one point $\pi(p) \in S$ via some trajectory. We fix any local coordinate system (w, x) on S and then a local coordinate system (w, x, y) on U such that the values $w(p)$ and $x(p)$ are defined as $w(\pi(p))$ and $x(\pi(p))$, respectively, for each point $p \in U$, $y(p)$ is the oriented length $d^+(\pi(p), p)$ of the trajectory joining p with $\pi(p)$. Then $E_3 = \partial/\partial y$ can be extended in U to an orthonormal moving frame $\{E_1, E_2, E_3\}$. Let $\{\omega^1, \omega^2, \omega^3\}$ be the corresponding dual coframe. Then ω^i's are of the form

$$(1.1)\quad \begin{cases} \omega^i = a^i \mathrm{d}w + b^i \mathrm{d}x, \quad i = 1, 2, \\ \omega^3 = \mathrm{d}y + H\mathrm{d}w + G\mathrm{d}x. \end{cases}$$

The Ricci tensor $\hat{R}$ expressed with respect to $\{E_1, E_2, E_3\}$ has the form $\hat{R}_{ij} = \rho_i \delta_{ij}$. Because each ρ_i is expressed through the sectional curvature K_{ij} by the formula $\rho_i = \hat{R}_{ii} = \sum_{j \neq i} K_{ij}$, there exist a function $k = k(w, x, y)$ of the variables w, x and y, and a constant $\tilde{c}$ such that

$$(1.2)\quad \begin{cases} K_{12} = k, \quad K_{13} = K_{23} = \tilde{c}, \\ \rho_1 = \rho_2 = k + \tilde{c}, \quad \rho_3 = 2\tilde{c}. \end{cases}$$

Define now the components ω^i_j of the connection form by the standard formulas

$$(1.3)\quad \begin{cases} \mathrm{d}\omega^i - \sum_j \omega^j \wedge \omega^i_j = 0, \\ \omega^i_j + \omega^j_i = 0, \quad i, j = 1, 2, 3. \end{cases}$$

Because the Riemannian curvature tensor satisfies $R_{ijkl} = 0$ whenever at least three of the indices i, j, k and l are distinct, the formulas (1.2) are equivalent to

$$(1.4)\quad \begin{cases} \mathrm{d}\omega^1_2 + \omega^1_3 \wedge \omega^3_2 = k\, \omega^1 \wedge \omega^2, \\ \mathrm{d}\omega^1_3 + \omega^1_2 \wedge \omega^2_3 = \tilde{c}\, \omega^1 \wedge \omega^3, \\ \mathrm{d}\omega^2_3 + \omega^2_1 \wedge \omega^1_3 = \tilde{c}\, \omega^2 \wedge \omega^3. \end{cases}$$

Next, differentiate the equations (1.4) and substitute from (1.4). We obtain easily

$$(1.5)\quad \omega^1_3 \wedge \omega^1 \wedge \omega^2 = 0, \quad \omega^2_3 \wedge \omega^1 \wedge \omega^2 = 0$$

and

$$(1.6)\quad \mathrm{d}((k - \tilde{c})\, \omega^1 \wedge \omega^2) = 0.$$

The relations (1.5) mean that ω^1_3 and ω^2_3 are linear combinations of ω^1 and ω^2 only, and from the third equation of (1.3) it follows that $\mathrm{d}\omega^3$ is a multiple of $\omega^1 \wedge \omega^2$ *i.e.,* a multiple of $\mathrm{d}w \wedge \mathrm{d}x$. Then (1.1) implies that the functions G and H are independent of y.

Now, there is a local coordinate system $(\bar{w}, \bar{x}, y)$ (possibly in a smaller neighborhood of m) such that $\bar{w} = \bar{w}(w, x)$ and $\bar{x} = \bar{x}(w, x)$ are functions of w and x, and

$$(1.7)\quad \begin{cases} \omega^1 = P^1 \mathrm{d}\bar{w} + Q^1 \mathrm{d}\bar{x}, \\ \omega^2 = P^2 \mathrm{d}\bar{w} + Q^2 \mathrm{d}\bar{x}, \\ \omega^3 = \mathrm{d}y + \bar{H}(\bar{w}, \bar{x})\mathrm{d}\bar{w}. \end{cases}$$

Indeed, because the surface S is not orthogonal to the vector field E_3 at m, the Pfaffian form $H\mathrm{d}w + G\mathrm{d}x$ from (1.1) is nonzero in a neighborhood of m in M. Then we define $\bar{w} = \bar{w}(w, x)$ as a potential function of the Pfaffian equation $H\mathrm{d}w + G\mathrm{d}x = 0$, and the second function $\bar{x} = \bar{x}(w, x)$ can be defined as an arbitrary smooth function which is functionally independent of $\bar{w}$. In addition, there are new Pfaffian forms $\tilde{\omega}^1$ and $\tilde{\omega}^2$ such that $(\tilde{\omega}^1)^2 + (\tilde{\omega}^2)^2 = (\omega^1)^2 + (\omega^2)^2$ and $\tilde{\omega}^1$ does not involve the differential $\mathrm{d}\bar{x}$. We can summarize:

Proposition 1.1. *In a normal neighborhood of any point $m \in M$ there exist an orthonormal coframe $\{\omega^1, \omega^2, \omega^3\}$ and a local coordinate system (w, x, y) such that*

$$(1.8)\quad \begin{cases} \omega^1 = f\mathrm{d}w, \\ \omega^2 = A\mathrm{d}x + C\mathrm{d}w, \\ \omega^3 = \mathrm{d}y + H\mathrm{d}w. \end{cases}$$

Here f, A and C are smooth functions of the variables w, x and y, $fA \neq 0$, and H is a smooth function of the variables w and x.

The formula (1.6) can be now written in the form

$$(1.9)\quad ((k - \tilde{c})fA)'_y = 0, \quad i.e., \quad k - \tilde{c} = \frac{\sigma}{fA}$$

for some nonzero function $\sigma = \sigma(w, x)$.

Now, define the function $\chi = \chi(w, x, y)$ of the variables w, x and y by

$$(1.10)\quad \chi = \frac{1}{fA} = \frac{k - \tilde{c}}{\sigma}.$$

Then, using (1.8) and (1.10), we obtain easily the following expression for the components of the connection form:

$$(1.11)\quad \begin{cases} \omega_2^1 = -A\alpha\mathrm{d}x + R\mathrm{d}w + \beta\mathrm{d}y, \\ \omega_3^1 = A\beta\mathrm{d}x + S\,\mathrm{d}w, \\ \omega_3^2 = A'_y\mathrm{d}x + T\mathrm{d}w, \end{cases}$$

where

$$(1.12)\quad \begin{cases} \alpha = \chi(A'_w - C'_x - HA'_y), \\ \beta = \dfrac{\chi}{2}(H'_x + AC'_y - CA'_y) \end{cases}$$

and

$$(1.13)\quad \begin{cases} R = \chi f f'_x - C\alpha + H\beta, \\ S = f'_y + C\beta, \\ T = C'_y - f\beta. \end{cases}$$

The curvature conditions (1.4) then give a system of nine partial differential equations for our problem:

(A1) $(A\alpha)'_y + \beta'_x = 0,$

(A2) $R'_y - \beta'_w = 0,$

(A3) $(A\alpha)'_w + R'_x + SA'_y - A\beta T = -fAk,$

(B1) $A''_{yy} - A\beta^2 = -\tilde{c}A,$

(B2) $-A''_{yw} + T'_x + A(\beta R + \alpha S) = \tilde{c}AH,$

(B3) $T'_y - S\beta = -\tilde{c}C,$

(C1) $(A\beta)'_y + A'_y\beta = 0,$

(C2) $S'_x - (A\beta)'_w - (A\alpha T + A'_yR) = 0,$

(C3) $S'_y + T\beta = -\tilde{c}f.$

2 The first integrals and the reduction of the basic system of partial differential equations

The aim of this section is to replace the partial differential equations of the series (B) and (C) by a system of algebraic equations for the new functions depending only on the variables w and x.

First of all, we can eliminate (B2) and (C2) by the same procedure as in [11]: the equation (B2) is a consequence of (A1) and (B1); the equation (C2) is a consequence of (A1), (A2) and (C1). Moreover, Proposition 2.3 from [11] still holds (with a slight change of the notation). We have

Proposition 2.1. *The equations* (B3) *and* (C3) *are satisfied if and only if*

(2.1) $fT - CS = \varphi_0,$

where $\varphi_0 = \varphi_0(w, x)$ is an arbitrary function of the variables w and x. Moreover, we have, in the **hyperbolic case** $\tilde{c} = -\lambda^2$,

(2.2h) $S^2 + T^2 = \lambda[\varphi_1 \cosh(2\lambda y) + \varphi_2 \sinh(2\lambda y) - \varphi_3],$

(2.3h) $fS + CT = \varphi_2 \cosh(2\lambda y) + \varphi_1 \sinh(2\lambda y),$

(2.4h) $f^2 + C^2 = \dfrac{1}{\lambda}[\varphi_1 \cosh(2\lambda y) + \varphi_2 \sinh(2\lambda y) + \varphi_3],$

where the functions $\varphi_i = \varphi_i(w, x)$, $i = 1, 2, 3$, of the variables w and x satisfy the single relation

(2.5h) ${\varphi_0}^2 + {\varphi_2}^2 - ({\varphi_1}^2 - {\varphi_3}^2) = 0$

and in the **elliptic case** $\tilde{c} = \lambda^2$,

(2.2e) $S^2 + T^2 = \lambda[\varphi_1 \cos(2\lambda y) - \varphi_2 \sin(2\lambda y) + \varphi_3],$

(2.3e) $fS + CT = \varphi_2 \cos(2\lambda y) + \varphi_1 \sin(2\lambda y),$

(2.4e) $f^2 + C^2 = \frac{1}{\lambda}[-\varphi_1 \cos(2\lambda y) + \varphi_2 \sin(2\lambda y) + \varphi_3]$,

where the functions $\varphi_i = \varphi_i(w, x)$, $i = 1, 2, 3$, *of the variables* w *and* x *satisfy the single relation*

(2.5e) $\varphi_0{}^2 + \varphi_2{}^2 + \varphi_1{}^2 - \varphi_3{}^2 = 0$.

Proposition 2.2. *From the equations* (A1), (A2), (B1), (C1) *and* (C3)*, we have, in the hyperbolic case,*

(2.6h) $fA = f_1 \cosh(2\lambda y) + f_2 \sinh(2\lambda y) + f_3$

and, in the elliptic case,

(2.6e) $fA = f_1 \cos(2\lambda y) + f_2 \sin(2\lambda y) + f_3$,

where $f_i = f_i(w, x)$, $i = 1, 2, 3$, *are some functions of the variables* w *and* x.

There is a function $\varphi_4 = \varphi_4(w, x)$ *of the variables* w *and* x *such that, in the hyperbolic case,*

(2.7h) $SA = \lambda f_2 \cosh(2\lambda y) + \lambda f_1 \sinh(2\lambda y) + \varphi_4$

and, in the elliptic case,

(2.7e) $SA = \lambda f_2 \cos(2\lambda y) - \lambda f_1 \sin(2\lambda y) + \varphi_4$.

Further, the equation (A3) *is reduced to the equation*

(2.8) $(A\alpha)'_w + R'_x + \tau = 0$,

where

(2.9) $\tau = (SA)'_y + fA\rho_1$

is a function of the variables w *and* x.

Proof. From (C3) we obtain, using also (1.13),

(2.10) $(SA)'_y = SA'_y - A\beta T - \tilde{c}fA = f'_y A'_y + \beta(CA'_y - AC'_y) + f(A\beta^2 - \tilde{c}A)$.

Due to (B1) we obtain

(2.11) $(SA)'_y = f'_y A'_y + A''_{yy} f + \beta(CA'_y - AC'_y) = (A'_y f)'_y + \beta(CA'_y - AC'_y)$.

On the other hand, using (1.13) first and (C1) later, we get

(2.12) $(SA)'_y = [f'_y A + (A\beta)C]'_y = (f'_y A)'_y - \beta(CA'_y - AC'_y)$.

As the sum of (2.11) and (2.12) we obtain

(2.13) $2(SA)'_y = (fA'_y)'_y + (f'_y A)'_y = (fA)''_{yy}$.

Using (A1) and (A2), we obtain

(2.14) $[(A\alpha)'_w + R'_x]'_y = 0$.

Due to (2.10), (1.10) and $\rho_1 = k + \tilde{c}$, the equation (A3) takes in the form

(2.15) $(A\alpha)'_w + R'_x + (SA)'_y + fA\rho_1 = 0.$

According to (2.14), the function τ defined by (2.9) does not depend on y. This together with (2.15) implies (2.8). Also, the equations (2.13) and (2.9) imply

(2.16) $(fA)''_{yy} + 2fA\rho_1 = 2\tau.$

Substituting (1.10) and $\rho_1 = k + \tilde{c}$ into (2.16), we obtain

(2.17) $$\left(\frac{\sigma}{k-\tilde{c}}\right)''_{yy} + \frac{2(k+\tilde{c})\sigma}{k-\tilde{c}} - 2\tau = 0.$$

Because σ does not depend on y, putting

(2.18) $$F = \frac{1}{k-\tilde{c}} - \frac{\tau-\sigma}{2\tilde{c}\sigma},$$

we obtain

(2.19) $F''_{yy} + 4\tilde{c}F = 0.$

Moreover we get, from (2.18) and (1.10),

(2.20) $fA = F\sigma + f_3,$

where $f_3 = f_3(w, x)$ is an arbitrary function of the variables w and x.

The general solution of the partial differential equation (2.19) is, in the hyperbolic case,

(2.21h) $F = F_1 \cosh(2\lambda y) + F_2 \sinh(2\lambda y)$

and, in the elliptic case,

(2.21e) $F = F_1 \cos(2\lambda y) + F_2 \sin(2\lambda y),$

where $F_1 = F_1(w, x)$ and $F_2 = F_2(w, x)$ are arbitrary functions of the variables w and x. This together with (2.20) implies (2.6h) and (2.6e).

From (2.6he) and (2.13) we obtain (2.7he), respectively. ∎

Proposition 2.3. *The equation* (B1) *and* (C1) *are satisfied if and only if*

(2.22) $\beta A^2 = \lambda a_0,$

where $a_0 = a_0(w, x)$ is an arbitrary function and, moreover, we have
(a) *in the hyperbolic case,*

(2.23h) $A^2 = a_1 \cosh(2\lambda y) + a_2 \sinh(2\lambda y) + a_3,$

where $a_i = a_i(w, x)$, $i = 1, 2, 3$, are functions of the variables w and x satisfying

(2.24h) ${a_0}^2 + {a_2}^2 - ({a_1}^2 - {a_3}^2) = 0;$

(b) *in the elliptic case,*

(2.23e) $A^2 = a_1 \cos(2\lambda y) + a_2 \sin(2\lambda y) + a_3,$

where $a_i = a_i(w, x)$, $i = 1, 2, 3$, are functions of the variables w and x satisfying

(2.24e) ${a_0}^2 + {a_2}^2 + {a_1}^2 - {a_3}^2 = 0.$

The *proof* is the same as for Proposition 2.5 in [11] (with a slight change of the notation).

Proposition 2.4. *We have, in the hyperbolic case,*

$$\begin{aligned}2\lambda a_0AC &= [a_1\varphi_5 + 2\lambda(a_2f_3 - a_3f_2)]\cosh(2\lambda y)\\ &\quad + [a_2\varphi_5 - 2\lambda(a_3f_1 - a_1f_3)]\sinh(2\lambda y)\\ &\quad + a_3\varphi_5 - 2\lambda(a_2f_1 - a_1f_2)\end{aligned} \tag{2.25h}$$

and, in the elliptic case,

$$\begin{aligned}2\lambda a_0AC &= [a_1\varphi_5 + 2\lambda(a_2f_3 - a_3f_2)]\cos(2\lambda y)\\ &\quad + [a_2\varphi_5 + 2\lambda(a_3f_1 - a_1f_3)]\sin(2\lambda y)\\ &\quad + a_3\varphi_5 + 2\lambda(a_2f_1 - a_1f_2),\end{aligned} \tag{2.25e}$$

where $\varphi_5 = \varphi_5(w,x)$ is an arbitrary function of the variables w and x.

Proof. Subtracting equations (2.11) and (2.12), we get

$$(fA'_y - f'_yA)'_y + 2\beta(A'_yC - AC'_y) = 0,$$

that is,

$$(fA'_y - f'_yA)'_y = 2\beta A^2\frac{AC'_y - A'_yC}{A^2}.$$

Using (2.22), we get

$$(fA'_y - f'_yA)'_y = 2\lambda a_0\left(\frac{C}{A}\right)'_y. \tag{2.26}$$

Integrating (2.26) with respect to y and multiplying by A^3, we get

$$2\lambda a_0AC = \varphi_5A^2 + (fA)(A^2)'_y - A^2(fA)'_y, \tag{2.27}$$

where $\varphi_5 = \varphi_5(w,x)$ is an arbitrary function of the variables w and x. Substituting (2.6he) and (2.23he) into (2.27), we obtain our assertion, respectively. ∎

The following proposition is more explicit.

Proposition 2.5. *We have, in the hyperbolic case,*

$$AC = b_1\cosh(2\lambda y) + b_2\sinh(2\lambda y) + b_3 \tag{2.28h}$$

and, in the elliptic case,

$$AC = b_1\cos(2\lambda y) + b_2\sin(2\lambda y) + b_3, \tag{2.28e}$$

where $b_i = b_i(w,x)$, $i = 1,2,3$, are functions of the variables w and x.

Proof. For $a_0 \neq 0$, the assertion (2.28he) is a direct consequence of (2.25he), respectively.

Suppose now $\tilde{c} = \epsilon\lambda^2$, $\epsilon = \pm 1$, and $a_0 = 0$. Then $\beta = 0$ by (2.22) and we get from $(1.13)_3$ and (B3) that

$$C''_{yy} = -\tilde{c}\, C = -\epsilon\lambda^2 C.$$

Hence we get, in the hyperbolic case,

(2.29h) $C = r\cosh(\lambda y) + s\sinh(\lambda y)$

and, in the elliptic case,

(2.29e) $C = r\cos(\lambda y) + s\sin(\lambda y)$,

where $r = r(w, x)$ and $s = s(w, x)$ are arbitrary functions of the variables w and x. On the other hand, (2.23he) and (2.24he) with $a_0 = 0$ imply, in the hyperbolic case,

(2.30h) $A = p\cosh(\lambda y) + q\sinh(\lambda y)$

and, in the elliptic case,

(2.30e) $A = p\cos(\lambda y) + q\sin(\lambda y)$

with some functions $p = p(w, x)$ and $q = q(w, x)$ of the variables w and x. Hence (2.28he) follows. ∎

Remark. We denote $\operatorname{sgn}\tilde{c}$ by ϵ in the sequel. This notation will be used later to unify many formulas for the hyperbolic and the elliptic case.

Now we introduce the function $h = h(w, x)$ by

(2.31) $h = H'_x$.

Proposition 2.6. *We have*

$$(2.32)\quad \begin{cases} ha_1 = 2\lambda[a_0 f_1 + a_2 b_3 - a_3 b_2], \\ ha_2 = 2\lambda[a_0 f_2 + \epsilon(a_3 b_1 - a_1 b_3)], \\ ha_3 = 2\lambda[a_0 f_3 - (a_1 b_2 - a_2 b_1)]. \end{cases}$$

Proof. From $(1.12)_2$ we get

$$h = 2fA\beta - (AC)'_y + 2A'_y C.$$

Then (2.22) and (1.10) imply

(2.33) $hA^2 = 2\lambda a_0 f A - A^2(AC)'_y + (AC)(A^2)'_y$.

Now we use (2.6he), (2.23he) and (2.28he) to get (2.32he). ∎

From (2.21he), (1.10) and (2.1) we obtain

(2.34) $S = f\chi Q, \qquad T = C\chi Q + \varphi_0\chi A,$

where, in the hyperbolic case,

(2.35h) $Q = \lambda f_2 \cosh(2\lambda y) + \lambda f_1 \sinh(2\lambda y) + \varphi_4$

and, in the elliptic case,

(2.35e) $Q = \lambda f_2 \cos(2\lambda y) - \lambda f_1 \sin(2\lambda y) + \varphi_4.$

Substituting from (2.34) into the partial differential equation (C3), we obtain, using also (2.22),

$$\left(f\chi Q'_y - \frac{A'_y}{A^2} Q\right) A^2 + \lambda a_0 C\chi Q + \lambda a_0 \varphi_0 \chi A = -\tilde{c} f A^2.$$

Multiplying this equation by A and using (2.27) and (1.10), we get

(2.36) $2fAQ'_y + \varphi_5 Q - Q(fA)'_y + 2\lambda a_0 \varphi_0 + 2\tilde{c}(fA)^2 = 0.$

Substituting from (2.6he) and (2.35he) into (2.36), we obtain

(2.37) $\begin{cases} f_1(\varphi_5 - 2\varphi_4) = 0, \quad f_2(\varphi_5 - 2\varphi_4) = 0, \\ \varphi_4\varphi_5 + 2\lambda a_0 \varphi_0 - 2\lambda^2[{f_2}^2 + \epsilon({f_1}^2 - {f_3}^2)] = 0. \end{cases}$

Substituting (2.35he) into (2.34), we obtain, in the hyperbolic case,

(2.38h) $S = f\chi[\lambda f_2 \cosh(2\lambda y) + \lambda f_1 \sinh(2\lambda y) + \varphi_4],$

(2.39h) $T = C\chi[\lambda f_2 \cosh(2\lambda y) + \lambda f_1 \sinh(2\lambda y) + \varphi_4] + \varphi_0\chi A$

and, in the elliptic case,

(2.38e) $S = f\chi[\lambda f_2 \cos(2\lambda y) - \lambda f_1 \sin(2\lambda y) + \varphi_4],$

(2.39e) $T = C\chi[\lambda f_2 \cos(2\lambda y) - \lambda f_1 \sin(2\lambda y) + \varphi_4] + \varphi_0\chi A.$

Hence we obtain, in the hyperbolic case,

(2.40h) $fA(CT + fS) = \varphi_0 AC + [\lambda f_2 \cosh(2\lambda y) + \lambda f_1 \sinh(2\lambda y) + \varphi_4](f^2 + C^2)$

and, in the elliptic case,

(2.40e) $fA(CT + fS) = \varphi_0 AC + [\lambda f_2 \cos(2\lambda y) - \lambda f_1 \sin(2\lambda y) + \varphi_4](f^2 + C^2).$

Substituting (2.3he), (2.4he) and (2.6he) into (2.40he), we get in the hyperbolic case,

(2.41h)
$$\begin{aligned} \varphi_0 AC = {} & (f_3\varphi_2 - f_2\varphi_3 - \frac{1}{\lambda}\varphi_1\varphi_4)\cosh(2\lambda y) \\ & + (f_3\varphi_1 - f_1\varphi_3 - \frac{1}{\lambda}\varphi_2\varphi_4)\sinh(2\lambda y) \\ & + f_1\varphi_2 - f_2\varphi_1 - \frac{1}{\lambda}\varphi_3\varphi_4 \end{aligned}$$

and, in the elliptic case,

$$\varphi_0 AC = (f_3\varphi_2 - f_2\varphi_3 + \frac{1}{\lambda}\varphi_1\varphi_4)\cos(2\lambda y) + (f_3\varphi_1 + f_1\varphi_3 - \frac{1}{\lambda}\varphi_2\varphi_4)\sin(2\lambda y) + f_1\varphi_2 + f_2\varphi_1 - \frac{1}{\lambda}\varphi_3\varphi_4. \tag{2.41e}$$

Another consequence of (2.38he) and (2.39he) is, in the hyperbolic case,

$$\begin{aligned}(fA)^2(S^2+T^2) &= [\lambda^2 f_2{}^2\cosh^2(2\lambda y) + \lambda^2 f_1{}^2\sinh^2(2\lambda y) + 2\lambda^2 f_1 f_2\cosh(2\lambda y)\sinh(2\lambda y)\\ &\quad + 2\lambda f_2\varphi_4\cosh(2\lambda y) + 2\lambda f_1\varphi_4\sinh(2\lambda y) + \varphi_4{}^2](f^2+C^2)\\ &\quad + 2\varphi_0 AC[\lambda f_2\cosh(2\lambda y) + \lambda f_1\sinh(2\lambda y) + \varphi_4] + \varphi_0{}^2 A^2\end{aligned} \tag{2.42h}$$

and, in the elliptic case,

$$\begin{aligned}(fA)^2(S^2+T^2) &= [\lambda^2 f_2{}^2\cos^2(2\lambda y) + \lambda^2 f_1{}^2\sin^2(2\lambda y) - 2\lambda^2 f_1 f_2\cos(2\lambda y)\sin(2\lambda y)\\ &\quad + 2\lambda f_2\varphi_4\cos(2\lambda y) - 2\lambda f_1\varphi_4\sin(2\lambda y) + \varphi_4{}^2](f^2+C^2)\\ &\quad + 2\varphi_0 AC[\lambda f_2\cos(2\lambda y) - \lambda f_1\sin(2\lambda y) + \varphi_4] + \varphi_0{}^2 A^2.\end{aligned} \tag{2.42e}$$

Using the formulas (2.2he), (2.4he), (2.6he), (2.23he) and (2.41he), we obtain from (2.42he)

$$\begin{cases}\lambda\varphi_0{}^2 a_1 = \varphi_1[\lambda^2(f_1{}^2 - \epsilon f_2{}^2 + f_3{}^2) - \epsilon\varphi_4{}^2]\\ \qquad\qquad + 2\lambda^2 f_1(\epsilon f_3\varphi_3 - f_2\varphi_2) + 2\lambda\varphi_4(f_2\varphi_3 - f_3\varphi_2),\\ \lambda\varphi_0{}^2 a_2 = \epsilon\varphi_2[\lambda^2(f_1{}^2 - \epsilon f_2{}^2 - f_3{}^2) + \epsilon\varphi_4{}^2]\\ \qquad\qquad + 2\lambda^2 f_2(f_1\varphi_1 + \epsilon f_3\varphi_3) - 2\lambda\varphi_4(f_3\varphi_1 + \epsilon f_1\varphi_3),\\ \lambda\varphi_0{}^2 a_3 = \epsilon\varphi_3[\lambda^2(f_1{}^2 + \epsilon f_2{}^2 + f_3{}^2) + \epsilon\varphi_4{}^2]\\ \qquad\qquad + 2\lambda^2 f_3(f_1\varphi_1 - f_2\varphi_2) - 2\lambda\varphi_4(f_1\varphi_2 + \epsilon f_2\varphi_1).\end{cases} \tag{2.43}$$

Consider now the identity $(AC)^2 = A^2(f^2 + C^2) - (Af)^2$. Substituting from (2.4he), (2.6he), (2.23he) and (2.28he), we get a system of quadratic equations

$$\begin{cases}\lambda(b_1{}^2 - \epsilon b_2{}^2 + f_1{}^2 - \epsilon f_2{}^2) = -\epsilon(a_1\varphi_1 + a_2\varphi_2),\\ \lambda(b_1{}^2 + \epsilon b_2{}^2 + 2b_3{}^2 + f_1{}^2 + \epsilon f_2{}^2 + 2f_3{}^2) = -\epsilon(a_1\varphi_1 - a_2\varphi_2) + 2a_3\varphi_3,\\ 2\lambda(b_1 b_2 + f_1 f_2) = a_1\varphi_2 - \epsilon a_2\varphi_1,\\ 2\lambda(b_1 b_3 + f_1 f_3) = a_1\varphi_3 - \epsilon a_3\varphi_1,\\ 2\lambda(b_2 b_3 + f_2 f_3) = a_2\varphi_3 + a_3\varphi_2.\end{cases} \tag{2.44}$$

In the notation (2.28he) we can rewrite (2.25he) in the form

$$(2.45)\quad \begin{cases} 2\lambda a_0 b_1 = a_1\varphi_5 + 2\lambda(a_2 f_3 - a_3 f_2), \\ 2\lambda a_0 b_2 = a_2\varphi_5 + 2\epsilon\lambda(a_3 f_1 - a_1 f_3), \\ 2\lambda a_0 b_3 = a_3\varphi_5 - 2\lambda(a_1 f_2 - a_2 f_1). \end{cases}$$

Also, we can rewrite (2.41he) in the form

$$(2.46)\quad \begin{cases} \lambda\varphi_0 b_1 = -\lambda(f_2\varphi_3 - f_3\varphi_2) + \epsilon\varphi_1\varphi_4, \\ \lambda\varphi_0 b_2 = \lambda(f_3\varphi_1 + \epsilon f_1\varphi_3) - \varphi_2\varphi_4, \\ \lambda\varphi_0 b_3 = \lambda(f_1\varphi_2 + \epsilon f_2\varphi_1) - \varphi_3\varphi_4. \end{cases}$$

Proposition 2.7. *If* $a_0 \neq 0$, *then we have*

$$(2.47)\quad h = -\frac{2\lambda[\epsilon(a_1 f_1 - a_3 f_3) + a_2 f_2]}{a_0}.$$

Proof. The assertion follows from (2.32), (2.45) and (2.24he). ∎

Now we have the main results of this section.

Theorem 2.8. *Let* λ *be a positive constant. Let* $\varphi_0, \varphi_1, \ldots, \varphi_5, a_0, a_1, a_2, a_3, b_1, b_2, b_3, f_1, f_2, f_3$ *and* h *be functions of two variables* w *and* x *defined in some domain* $V \subset \mathbb{R}^2(w,x)$*, satisfying eight collections of algebraic equations* (2.5), (2.24), (2.32), $(2.37)_2$, (2.43), (2.44), (2.45) *and* (2.46) (*either of hyperbolic type, or of elliptic type*) *with the corresponding parameter* λ*, and such that* ${a_1}^2 + {a_2}^2 + {a_3}^2 > 0$ *in* V.

Let A*,* f*,* C *and* H *be functions defined in a domain* $U \subset \mathbb{R}^3(w,x,y)$*, where* $A \neq 0$*, by the formulas* (2.23), (2.6), (2.28) *and* (2.31) *of the corresponding type, and let the metric* g *be defined on* U *by* (1.8)*. Further, let* α*,* β *and* R *be defined as in* $(1.12)_1$, (2.22), $(1.13)_1$*. Then the curvature conditions* (1.4) *are satisfied for some function* $k = k(w,x,y)$ *of the variables* w*,* x *and* y*, and for the corresponding constant* $\tilde{c} = \pm\lambda^2$ *if and only if the system of partial differential equations* (A1) *and* (A2) *is satisfied.*

Proof. The assertion follows from the whole series of propositions and formulas given in this section. ∎

Remark. Because we *do not prescribe* the function $k = k(w,x,y)$ in advance, the equation (A3) (or, equivalently, (2.8)) does not give any additional condition. But, due to (2.9) and (1.2), the equation (2.8) can be considered just as a formula for calculating the Ricci eigenvalue ρ_1 or the scalar curvature $\mathrm{Sc}(g) = 2k + 4\tilde{c}$ of (M,g).

Remark. The algebraic conditions mentioned above are, of course, far from being independent, but they are all useful.

We conclude this section by proving additional algebraic equations between our basic functions.

Proposition 2.9. *We have*

(2.48) $\varphi_5 = 2\varphi_4,$

(2.49) $\varphi_0 A^2 - \lambda a_0(f^2 + C^2) + \varphi_5 AC + hfA = 0.$

Proof. If ${f_1}^2 + {f_2}^2 \neq 0$, then (2.48) follows from $(2.37)_{1,2}$. If $f_1 = f_2 = 0$, then we proceed as in the proof of Proposition 4.1 in [11].

To derive (2.49), we rewrite (2.37) using (2.48) in the form

(2.50) $\lambda a_0 \varphi_0 = \lambda^2[{f_2}^2 + \epsilon({f_1}^2 - {f_3}^2)] - {\varphi_4}^2.$

Suppose $a_0 \neq 0$. Then (2.45) and (2.48) imply

$$(2.51)\quad \begin{cases} b_1 = \dfrac{a_1\varphi_4 + \lambda(a_2 f_3 - a_3 f_2)}{\lambda a_0}, \\ b_2 = \dfrac{a_2\varphi_4 + \epsilon\lambda(a_3 f_1 - a_1 f_3)}{\lambda a_0}, \\ b_3 = \dfrac{a_3\varphi_4 - \lambda(a_1 f_2 - a_2 f_1)}{\lambda a_0}. \end{cases}$$

Now we substitute for A^2, $f^2 + C^2$, AC, φ_5, h and fA of the left-hand side of (2.49) from (2.23he), (2.4he), (2.28he), (2.48) and (2.6he), respectively. Then the identity (2.49) follows. If $a_0 = 0$, we use the direct check as in [11]. ∎

Proposition 2.10. *The following algebraic formulas hold*

(2.52) $2\lambda(a_1 f_1 + \epsilon a_2 f_2 - a_3 f_3) = -\epsilon a_0 h,$

(2.53) $4\lambda^2(b_1 f_1 + \epsilon b_2 f_2 - b_3 f_3) = -\epsilon\varphi_5 h,$

(2.54) $2\lambda(\varphi_1 f_1 - \varphi_2 f_2 - \varphi_3 f_3) = \epsilon\varphi_0 h,$

(2.55) $2\lambda(a_1 b_1 + \epsilon a_2 b_2 - a_3 b_3) = -\epsilon a_0 \varphi_5.$

Proof. From (2.24he) and (2.32) we obtain

$$2\lambda a_0(a_1 f_1 + \epsilon a_2 f_2 - a_3 f_3) = -\epsilon {a_0}^2 h.$$

Hence we obtain (2.52) if $a_0 \neq 0$. From (2.45) and (2.24he) we obtain

$$2\lambda a_0(b_1 f_1 + \epsilon b_2 f_2 - b_3 f_3) = \varphi_5(a_1 f_1 + \epsilon a_2 f_2 - a_3 f_3),$$

which together with (2.52) implies (2.53) when $a_0 \neq 0$. From (2.46) we obtain

$$\varphi_4(\varphi_1 f_1 - \varphi_2 f_2 - \varphi_3 f_3) = -\lambda\varphi_0(b_1 f_1 + b_2 f_2 - b_3 f_3),$$

hence, if $a_0\varphi_4 \neq 0$, we obtain (2.54) using (2.53) and (2.48). Finally from (2.45) we obtain

$$2\lambda a_0(a_1 b_1 + \epsilon a_2 b_2 - a_3 b_3) = -\epsilon {a_0}^2 \varphi_5.$$

Thus we obtain (2.55) when $a_0\varphi_4 \neq 0$.

For $a_0\varphi_4 = 0$ we use the continuity argument or a rather lengthy direct check (*cf.* Proposition 4.10 in [10]). ∎

3 The Riemannian invariants

Let (M, g) be given locally as in Proposition 1.1. We rewrite the formulas (1.11) using the forms ω^1, ω^2 and ω^3 as a basis. It follows

$$(3.1)\quad \begin{cases} \omega^1_2 = \chi f'_x\,\omega^1 - \alpha\,\omega^2 + \beta\,\omega^3, \\ \omega^1_3 = \dfrac{f'_y}{f}\,\omega^1 + \beta\,\omega^2, \\ \omega^2_3 = (\beta - h\chi)\,\omega^1 + \dfrac{A'_y}{A}\,\omega^2, \quad h = H'_x. \end{cases}$$

We also write, for brevity,

$$(3.2)\quad \omega^1_3 = a\,\omega^1 + b\,\omega^2, \quad \omega^2_3 = c\,\omega^1 + e\,\omega^2,$$

where

$$(3.3)\quad a = \frac{f'_y}{f}, \quad b = \beta, \quad c = \beta - h\chi, \quad e = \frac{A'_y}{A}.$$

Using the standard formula $\nabla_{E_j} E_i = \sum_k \omega^k_i(E_j)E_k$, $i, j = 1, 2, 3$, from [8], we obtain

$$(3.4)\quad \begin{cases} \nabla_{E_1} E_1 = -\chi f'_x E_2 - aE_3, & \nabla_{E_1} E_2 = \chi f'_x E_1 - cE_3, \\ \nabla_{E_2} E_1 = \alpha E_2 - bE_3, & \nabla_{E_2} E_2 = -\alpha E_1 - eE_3, \\ \nabla_{E_1} E_3 = aE_1 + cE_2, & \nabla_{E_2} E_3 = bE_1 + eE_2, \\ \nabla_{E_3} E_1 = -bE_2, & \nabla_{E_3} E_2 = bE_1, \\ \nabla_{E_3} E_3 = 0. \end{cases}$$

The last formula shows that the trajectories of the unit vector field E_3 (consisting of the eigenvectors of the Ricci tensor $\hat{R}$ corresponding to $\rho_3 = 2\tilde{c}$) are geodesics.

For the Ricci tensor $\hat{R}$ we get, using the notation (1.2) and the adapted local orthonormal coframe $\{\omega^1, \omega^2, \omega^3\}$,

$$(3.5)\quad \hat{R} = (k + \tilde{c})(\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2) + 2\tilde{c}\,(\omega^3 \otimes \omega^3).$$

Using (3.1), (3.2) and the standard formula $\nabla_X \omega^i = -\sum_j \omega^i_j(X)\omega^j$, we obtain

$$(3.6)\quad \begin{aligned} \nabla\hat{R} = {} & \mathrm{d}k \otimes (\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2) \\ & + (\tilde{c} - k)\{(a\,\omega^1 + b\,\omega^2) \otimes (\omega^1 \otimes \omega^3 + \omega^3 \otimes \omega^1) \\ & \qquad + (c\,\omega^1 + e\,\omega^2) \otimes (\omega^2 \otimes \omega^3 + \omega^3 \otimes \omega^2)\}, \end{aligned}$$

where a, b, c and e are given by (3.3). Hence we also get

$$(3.7)\quad \begin{aligned} ||\nabla\hat{R}||^2 &= 2||\mathrm{d}k||^2 + 2(\tilde{c} - k)^2(a^2 + b^2 + c^2 + e^2) \\ &= 2||\mathrm{d}\rho_1||^2 + 2(\rho_1 - \rho_3)^2(a^2 + b^2 + c^2 + e^2). \end{aligned}$$

Because $\hat{R}$ is a Riemannian invariant tensor, $\nabla\hat{R}$ is an invariant tensor. Also, because $E_3 = \partial/\partial y$ is uniquely determined by the geometry of (M, g) up to sign, $\omega^3 \otimes \omega^3$ is an invariant tensor. Hence we see from (3.5) and (3.6) that the tensor

$$\begin{aligned} Q &= (a\,\omega^1 + b\,\omega^2) \otimes (\omega^1 \otimes \omega^3 + \omega^3 \otimes \omega^1) \\ &\quad + (c\,\omega^1 + e\,\omega^2) \otimes (\omega^2 \otimes \omega^3 + \omega^3 \otimes \omega^2) \end{aligned} \tag{3.8}$$

is also invariant. Now because E_1 and E_2 are determined up to an orthogonal transformation (with functional coefficients), the functions

$$\begin{cases} Q(E_1, E_1, E_3) + Q(E_2, E_2, E_3) = a + e, \\ Q(E_2, E_1, E_3) - Q(E_1, E_2, E_3) = b - c \end{cases} \tag{3.9}$$

are Riemannian invariants up to sign.

The square of the norm $\|Q\|^2 = 2(a^2 + b^2 + c^2 + e^2)$ is a Riemannian invariant and hence (equivalently) $ae - bc$ is a Riemannian invariant. We summarize:

Proposition 3.1. *The function $ae - bc$ is a Riemannian invariant, and $a + e$ and $b - c$ are Riemannian invariants up to sign (i.e., depending on the orientation of the principal geodesics). Further, the partial derivative of any Riemannian invariant with respect to y is a Riemannian invariant up to sign.*

Using (1.10), we get

$$\begin{cases} a + e = (\ln(fA))'_y = -(\ln(k - \epsilon\lambda^2))'_y, \\ b - c = h\chi = \dfrac{h(k - \epsilon\lambda^2)}{\sigma}. \end{cases} \tag{3.10}$$

Further we have

$$ae - bc = \epsilon(2\lambda^2 f_3 \chi - \lambda^2). \tag{3.11}$$

The last formula is obtained by lengthy calculations using (2.52) and the obvious identities

$$(AA'_y)^2 + \lambda^2 {a_0}^2 = -\epsilon\lambda^2[(A^2 - a_3)^2 - a_3^2], \tag{3.12}$$

$$A^3 f'_y = (fA)'_y A^2 - (fA)(AA'_y). \tag{3.13}$$

Using (3.11) we see that, in the hyperbolic case,

$$\frac{fA}{f_3} = \frac{f_1 \cosh(2\lambda y) + f_2 \sinh(2\lambda y) + f_3}{f_3} \tag{3.14h}$$

is a Riemannian invariant and, in the elliptic case,

$$\frac{fA}{f_3} = \frac{f_1 \cos(2\lambda y) + f_2 \sin(2\lambda y) + f_3}{f_3} \tag{3.14e}$$

is a Riemannian invariant (assuming $f_3 \neq 0$ everywhere). (According to $(3.10)_2$, fA/h and f_3/h are then Riemannian invariants *up to sign* assuming $h \neq 0$ everywhere.)

Next, we give some simple results concerning isometry of Riemannian manifolds with the Ricci eigenvalues $\rho_1 = \rho_2$ and nonzero constant ρ_3 to be used later. Let (M, g) be such a manifold with the metric g given by (1.8) and let $(\bar{M}, \bar{g})$ be another such manifold with the metric $\bar{g}$ given by the orthonormal coframe

$$\bar{\omega}^1 = \bar{f}\mathrm{d}\bar{w}, \quad \bar{\omega}^2 = \bar{A}\mathrm{d}\bar{x} + \bar{C}\mathrm{d}\bar{w}, \quad \bar{\omega}^3 = \mathrm{d}\bar{y} + \bar{H}\mathrm{d}\bar{w}.$$

Suppose that there is an isometry $\Phi : (M, g) \longrightarrow (\bar{M}, \bar{g})$ given by

$$(3.15)\quad \bar{w} = \bar{w}(w, x, y), \quad \bar{x} = \bar{x}(w, x, y), \quad \bar{y} = \bar{y}(w, x, y).$$

Here we use $\bar{w}$, $\bar{x}$ and $\bar{y}$ as simple notations for $\bar{w} \circ \Phi$, $\bar{x} \circ \Phi$ and $\bar{y} \circ \Phi$, respectively, and we shall also write simply $\bar{\omega}^i$ instead of $\Phi^*\bar{\omega}^i$ for $i = 1, 2, 3$.

Propositions 5.2 and 5.3 from [10] still hold without change. We have:

Proposition 3.2. *The equation* (3.15) *can be reduced to the form*

$$\bar{w} = \bar{w}(w, x), \quad \bar{x} = \bar{x}(w, x), \quad \bar{y} = \varepsilon y + \phi(w, x), \quad \varepsilon = \pm 1,$$

where $\phi = \phi(w, x)$ is an arbitrary function of the variables w and x.

Proposition 3.3. *Suppose $\beta = a_0 = 0$ on (M, g) and $\bar{\beta} = \bar{a}_0 = 0$ on $(\bar{M}, \bar{g})$. Further, assume that $e^2 - a^2 \neq 0$ or $c \neq 0$ holds on (M, g). Then any isometry $\Phi : (M, g) \longrightarrow (\bar{M}, \bar{g})$ implies the equalities*

$$\bar{\omega}^i = \varepsilon_i\, \omega^i, \quad \varepsilon_i = \pm 1, \quad i = 1, 2, 3.$$

4 The asymptotic foliations and four types of spaces

Recall that the principal geodesics are trajectories of the vector field E_3. We introduce two basic definitions.

Definition 4.1. A smooth surface $N \subset (M, g)$ is called an *asymptotic leaf* if it is generated by the principal geodesics and its tangent planes are parallel along these principal geodesics with respect to the Levi-Civita connection ∇ of (M, g).

Definition 4.2. An *asymptotic distribution* on M is a two-dimensional distribution which is integrable and whose integral manifolds are asymptotic leaves. The integral manifolds of an asymptotic distribution determine a foliation of M, which is called an *asymptotic foliation*.

Let N be an asymptotic leaf. Then its tangent planes along N can be described by

$$(4.1)\quad \sin\varphi\, \omega^1 + \cos\varphi\, \omega^2 = 0,$$

where φ is a smooth function on N. By the same argument as in [10], we see that, according to the integrability condition and the asymptoticity condition, the tangent distribution of N satisfies

$$(4.2)\quad b\, \sin^2\varphi + (e - a)\cos\varphi \sin\varphi - c\, \cos^2\varphi = 0$$

and hence

$$(4.3)\quad c(\omega^1)^2 + (e - a)\omega^1\omega^2 - b(\omega^2)^2 = 0,$$

where a, b, c and e are given by (3.3). Of course, an asymptotic distribution must satisfy these equations locally on the whole of M. Conversely, *any smooth distribution satisfying* (4.3) *is an asymptotic distribution*. For the details of the proof see Section 6 in [10].

The following proposition is almost obvious.

Proposition 4.3. *Let* $\Delta = (e-a)^2 + 4bc$ *be the discriminant of the quadratic equation* (4.3). *Then we have*:

(E) *If* $\Delta < 0$ *on* (M, g)*, then there is no real asymptotic distribution on* M.

(H) *If* $\Delta > 0$ *on* (M, g)*, then there are exactly two different asymptotic distributions on* M.

(P) *If* $\Delta = 0$ *on* (M, g) *and some of the functions* $e-a$*,* b *and* c *are nonzero at each point, then there is a unique asymptotic distribution on* M.

(Pℓ) *If* $e - a = b = c = 0$ *on* M*, then any* π*-projectable smooth two-dimensional distribution on* M *is asymptotic, where* π *is the projection* $\pi : (w, x, y) \longmapsto (w, x)$.

Definition 4.4. A space (M, g) is said to be of *type* (E), (H), (P) or (Pℓ), respectively, (called also *elliptic*, *hyperbolic*, *parabolic* and *planar* type, respectively) if the corresponding case of Proposition 4.3 holds on the whole of M.

Corollary 4.5. *The space* (M, g) *is of type* (Pℓ) *if and only if* $f = \xi A$*,* $C = \zeta A$ *and* $\beta = 0$*, where* $\xi = \xi(w, x) \neq 0$ *and* $\zeta = \zeta(w, x)$ *are arbitrary functions of the variables* w *and* x*. Assuming* $\beta = 0$*,* (M, g) *is of type* (P) *if and only if* $f = \xi A$ *and* $h \neq 0$.

Proof. The relation $e - a = 0$ means $(f/A)'_y = 0$, $b = 0$ means $\beta = 0$, and $c = 0$ means $h = 0$. Due to $(1.12)_2$, $\beta = h = 0$ means $(C/A)'_y = 0$. Hence the first part of the Proposition follows. Further, if (M, g) is of type (P), then (4.3) with $b = 0$ must be reduced to $c(\omega^1)^2 = 0$, *i.e.*, $e - a = 0$ and $h = -c \neq 0$. Hence the second part follows. ∎

In the following two sections we are going to describe explicitly all spaces of all types (E), (H), (P) and (Pℓ) in both hyperbolic case ($\tilde{c} < 0$) and elliptic case ($\tilde{c} > 0$). Because we are interested in the local classification, we investigate only the "pure" cases and not the combined ones in the sequel. (For a global treatment of *some* of our geometric types see [21]).

We add some more details:

Proposition 4.6. *The equation* (4.3) *is equivalent with the equation*

$$\lambda a_0 \mathrm{d}x^2 + \varphi_5 \mathrm{d}x\mathrm{d}w - \varphi_0 \mathrm{d}w^2 = 0. \tag{4.4}$$

Proof. We can apply the same procedure as in [10] (Proof of Theorem 6.5). Here we use formulas (2.49) and (2.27) for this purpose. ∎

Hence we can decide about the type of the space (M, g) according to the following

Proposition 4.7. *Let* $\Delta' = {\varphi_5}^2 + 4\lambda a_0 \varphi_0$ *be the discriminant of the quadratic equation* (4.4). *Then the analogy of Proposition* 4.3 *holds if* Δ *is replaced by* Δ'.

Proof. One can show easily that $\Delta' = (fA)^2\Delta$. ∎

Also, notice that Δ' is given alternatively by the formula

(4.5) $\Delta' = 4\lambda^2[f_2{}^2 + \epsilon(f_1{}^2 - f_3{}^2)]$.

Indeed, combining $(2.37)_3$ with (2.48), we obtain at once

(4.6) $\varphi_5{}^2 + 4\lambda a_0\varphi_0 - 4\lambda^2[f_2{}^2 + \epsilon(f_1{}^2 - f_3{}^2)] = 0$.

Now, the following Theorem will be crucial for the explicit geometric classification of the manifolds of types (H), (P) and (Pℓ) in Section 5. Its proof is analogous to that of Theorem 6.6 from [10].

Theorem 4.8. *For each manifold of types* (H) *and* (P)*, there exists a transformation of local coordinates preserving the form* (1.8) *of the metric and annihilating the functions* β *and* a_0.

Remark. As concerns the type (Pℓ), we have $\beta = 0$ and hence $a_0 = 0$ (in a neighborhood of p) by definition. Thus for every space (M, g) of type (H), (P) or (Pℓ) we can assume $\beta = a_0 = 0$. Conversely, from (4.3) or (4.4) we see that $\beta = a_0 = 0$ always implies that (M, g) is one of the types (H), (P) and (Pℓ).

In this final part of this Section we prove some geometric results on asymptotic foliations. Propositions 6.10 and 6.11 from [10] still hold without change. We have:

Proposition 4.9. *If* $h = 0$*, then* (M, g) *is of type* (H), (P) *or* (Pℓ)*. On a space of type* (H)*,* $h = 0$ *means that the asymptotic foliations* $\mathscr{F}_1$ *and* $\mathscr{F}_2$ *are mutually orthogonal.*

Proof. The relation $h = 0$ means $b = c$ and hence $(e - a)^2 + 4bc \geq 0$. In the type (H), the equation (4.2) means $2b\cos(2\varphi) + (a - e)\sin(2\varphi) = 0$. Hence if φ characterizes one of the asymptotic foliations, then $\varphi + \pi/2$ characterizes the second one. From (4.1) we see that both foliations are mutually orthogonal. ∎

Proposition 4.10. *Let the metric* g *be of one of the types* (H), (P) *and* (Pℓ) *expressed in such a coordinate system that* $\beta = a_0 = 0$*. If* $\alpha = 0$*, then at least one of the asymptotic foliations is totally geodesic.*

Proof. Because $b = 0$, formulas (3.4) show that $\mathrm{span}\{E_2, E_3\}$ is an asymptotic distribution. But the corresponding asymptotic foliation is totally geodesic if and only if $\nabla_{E_2}E_2 \in \mathrm{span}\{E_2, E_3\}$, that is $\alpha = 0$. ∎

5 The explicit classification of asymptotically foliated spaces

In this section we shall explicitly classify all spaces of types (H), (P) and (Pℓ). Moreover, we shall answer the question *how the distinct locally isometry classes can be parameterized. In this section we always assume* $\beta = a_0 = 0$, which is allowed by Theorem 4.8 (see Remark 4).

We shall start with some general results.

Proposition 5.1. *For types* (H), (P) *and* (Pℓ)*, the coefficients A, C and f from* (1.8) *can be expressed, in the hyperbolic case, by*

$$(5.1h)\quad \begin{cases} A = p\cosh(\lambda y) + q\sinh(\lambda y), \\ C = r\cosh(\lambda y) + s\sinh(\lambda y), \\ f = t\cosh(\lambda y) + u\sinh(\lambda y) \end{cases}$$

and, in the elliptic case, by

$$(5.1e)\quad \begin{cases} A = p\cos(\lambda y) + q\sin(\lambda y), \\ C = r\cos(\lambda y) + s\sin(\lambda y), \\ f = t\cos(\lambda y) + u\sin(\lambda y), \end{cases}$$

where p, q, r, s, t and u are functions of the variables w and x such that

$$(5.2)\quad \lambda(qr - ps) = h.$$

Moreover, if $h \neq 0$*, we may assume* $h = 1$ *and* $H = x$*, and if* $h = 0$ *on an open subset, we may assume* $H = 0$ *on this subset.*

Proof. Because $\beta = a_0 = 0$, the equation (B1) implies $A''_{yy} = -\epsilon\lambda^2 A$, and the equations (B3) and (C3) together with (1.13) imply $C''_{yy} = -\epsilon\lambda^2 C$ and $f''_{yy} = -\epsilon\lambda^2 f$, respectively. The formula (5.2) follows from $(1.12)_2$ and (5.1he) because $h = H'_x$ and $\beta = 0$.

It remains to prove the last part. If $h \neq 0$, then $H'_x \neq 0$ and one can introduce the new variable $\bar{x} = H(w, x)$ instead of x. Then we get our orthonormal coframe in the standard form

$$\omega^1 = f\,\mathrm{d}w, \quad \omega^2 = \frac{A}{h}\,\mathrm{d}\bar{x} + \left(C - \frac{AH'_w}{h}\right)\mathrm{d}w, \quad \omega^3 = \mathrm{d}y + \bar{x}\,\mathrm{d}w.$$

Let now $h = 0$ on an open subset. Because H depends only on w, we get $\omega^3 = \mathrm{d}\bar{y}$, where $\bar{y} = y + \int H\,\mathrm{d}w$. ∎

In the sequel we put

$$(5.3)\quad \begin{cases} \mathcal{D} = p'_w - r'_x - \lambda q H, \\ \mathcal{E} = q'_w - s'_x + \epsilon\lambda p H. \end{cases}$$

(Thus $\mathcal{E}$ is different for the hyperbolic case and for the elliptic case.)

Proposition 5.2. *The differential equation* (A1) *is satisfied if and only if the following equation holds:*

$$(5.4)\quad u\,\mathcal{D} - t\,\mathcal{E} = 0.$$

Proof. Substituting from (5.1he) into (1.12) we get, in the hyperbolic case,

$$(5.5h)\quad A\alpha = \frac{A'_w - C'_x - HA'_y}{f} = \frac{\mathcal{D}\cosh(\lambda y) + \mathcal{E}\sinh(\lambda y)}{t\cosh(\lambda y) + u\sinh(\lambda y)}$$

and, in the elliptic case,

$$(5.5e)\quad A\alpha = \frac{A'_w - C'_x - HA'_y}{f} = \frac{\mathcal{D}\cos(\lambda y) + \mathcal{E}\sin(\lambda y)}{t\cos(\lambda y) + u\sin(\lambda y)},$$

where $\mathcal{D}$ and $\mathcal{E}$ are given by (5.3). Because $\beta = 0$, the equation (A1) simply means that $A\alpha$ does not depend on y and hence (5.4) follows. ∎

Proposition 5.3. *Assume that* (A1) *is satisfied and*

$$(5.6)\quad A\alpha = v, \quad v = v(w, x).$$

Then (A2) *is satisfied if and only if*

$$(5.7)\quad \lambda(qt'_x - pu'_x) = hv.$$

Proof. We have first, using $(1.13)_1$ and (5.6), in the hyperbolic case,

$$R = \chi f f'_x - C\alpha = \frac{f'_x - vC}{A}$$
$$= \frac{(t'_x - rv)\cosh(\lambda y) + (u'_x - sv)\sinh(\lambda y)}{p\cosh(\lambda y) + q\sinh(\lambda y)}$$

and, in the elliptic case,

$$R = \chi f f'_x - C\alpha = \frac{f'_x - vC}{A}$$
$$= \frac{(t'_x - rv)\cos(\lambda y) + (u'_x - sv)\sin(\lambda y)}{p\cos(\lambda y) + q\sin(\lambda y)}.$$

But the equation (A2) means that R does not depend on y, and (5.7) follows from (5.2). ∎

Now we can state the "converse" of Proposition 5.1.

Proposition 5.4. *Let p, q, r, s, t and u be arbitrary functions of the variables w and x. Define the functions A, C and f by* (5.1he)*, and let $H = H(w, x)$ be any function satisfying*

$$(5.8)\quad H'_x = h = \lambda(qr - ps).$$

If the equations (A1) *and* (A2) *are satisfied, then* (1.8) *defines a foliated metric of type* (H), (P) *or* (Pℓ).

Proof. Substituting from (5.1he) and (5.8) into $(1.12)_2$ we obtain $\beta = 0$. Hence, (B1) holds and (C1) is trivially satisfied. Since $(1.13)_{2,3}$ reduce to $S = f'_y$ and $T = C'_y$, respectively, (B3) and (C3) hold as a consequence of (5.1he). From Section 2 we know that the equations (B2) and (C2) are consequences of the other equations. As for the equation (A3), we can use Remark 2. Thus, the basic system of partial differential equations for the coefficients of (1.8) is reduced to the equations (A1) and (A2). Because $b = \beta = 0$, Proposition 4.3 shows that the metric must be of one of the type (H), (P) or (Pℓ). ∎

Because $b = \beta = 0$, the equation (4.3) for an asymptotic distribution reads

$$(5.9)\quad \omega^1(c\,\omega^1 + (e - a)\omega^2) = 0$$

or, equivalently, by (3.3) and $(1.12)_2$ together with $\beta = 0$,

$$(5.10)\quad \omega^1\left(\left(\frac{C}{A}\right)'_y \omega^1 - \left(\frac{f}{A}\right)'_y \omega^2\right) = 0.$$

We see that the equation $\omega^1 = 0$ defines an asymptotic distribution $\mathrm{span}\{E_2, E_3\}$, whose integral manifolds (the asymptotic leaves) are given by the equation $w =$ constant. From (3.4) it follows at once that this foliation is totally geodesic if and only if the function α from (1.12) vanishes identically. We distinguish two geometric situations on the spaces of types (H) and (P). We say a space of type (H) or (P) is *singular* or *generic* according to if the asymptotic distribution given by the equation $\omega^1 = 0$ is totally geodesic or not; or, equivalently, according to if $\alpha = 0$ or $\alpha \neq 0$.

5.1 The non-orthogonally foliated spaces of type (H)

Suppose now that the space (M, g) is of type (H) and the asymptotic foliations $\mathscr{F}_1$ and $\mathscr{F}_2$ are nowhere mutually orthogonal. From (5.10) we see that then necessarily $(f/A)'_y \neq 0$ and $(C/A)'_y \neq 0$. Using (5.1he) we see that this is equivalent with the inequalities $pu - qt \neq 0$ and $ps - qr \neq 0$.

Theorem 5.5. *The metric of a three-dimensional non-orthogonally foliated generic space of type* (H) *is locally determined, in the hyperbolic case, by an orthonormal coframe*

$$(5.11\mathrm{h})\quad \begin{cases} \omega^1 = [t\cosh(\lambda y) + u\sinh(\lambda y)]\mathrm{d}w, \\ \omega^2 = [p\cosh(\lambda y) + q\sinh(\lambda y)]\mathrm{d}x + [r\cosh(\lambda y) + s\sinh(\lambda y)]\mathrm{d}w, \\ \omega^3 = \mathrm{d}y + x\,\mathrm{d}w \end{cases}$$

and, in the elliptic case, by an orthonormal coframe

$$(5.11\mathrm{e})\quad \begin{cases} \omega^1 = [t\cos(\lambda y) + u\sin(\lambda y)]\mathrm{d}w, \\ \omega^2 = [p\cos(\lambda y) + q\sin(\lambda y)]\mathrm{d}x + [r\cos(\lambda y) + s\sin(\lambda y)]\mathrm{d}w, \\ \omega^3 = \mathrm{d}y + x\,\mathrm{d}w, \end{cases}$$

where p, q, r and s are arbitrary functions of the variables w and x such that $\lambda(ps - qr) = 1$, and t and u are calculated from p, q, r and s as follows:
Put (*as a special case of* (5.3))

$$(5.12)\quad \mathcal{D} = p'_w - r'_x - \lambda q x, \qquad \mathcal{E} = q'_w - s'_x + \epsilon\lambda p x.$$

If $\mathcal{E} \neq 0$, *then*

$$\text{(5.13a)} \quad \begin{cases} t = \dfrac{u\,\mathcal{D}}{\mathcal{E}}, \\ u = \exp\left(\dfrac{1}{2}\displaystyle\int P\,dx\right)\left[\displaystyle\int Q\exp\left(-\displaystyle\int P\,\mathrm{d}x\right)\mathrm{d}x\right]^{1/2}, \end{cases}$$

where

$$\text{(5.14a)} \quad P = \frac{2q(\mathcal{D}'_x\mathcal{E} - \mathcal{D}\mathcal{E}'_x)}{\mathcal{E}(q\,\mathcal{D} - p\,\mathcal{E})}, \quad Q = \frac{2\mathcal{E}^2}{\lambda(q\,\mathcal{D} - p\,\mathcal{E})}.$$

If $\mathcal{D} \neq 0$, *then*

$$\text{(5.13b)} \quad \begin{cases} t = \exp\left(\dfrac{1}{2}\displaystyle\int \bar{P}\,\mathrm{d}x\right)\left[\displaystyle\int \bar{Q}\exp\left(-\displaystyle\int \bar{P}\,\mathrm{d}x\right)\mathrm{d}x\right]^{1/2}, \\ u = \dfrac{t\,\mathcal{E}}{\mathcal{D}}, \end{cases}$$

where

$$\text{(5.14b)} \quad \bar{P} = \frac{2p(\mathcal{D}\mathcal{E}'_x - \mathcal{D}'_x\mathcal{E})}{\mathcal{D}(q\,\mathcal{D} - p\,\mathcal{E})}, \quad \bar{Q} = \frac{2\mathcal{D}^2}{\lambda(q\,\mathcal{D} - p\,\mathcal{E})}.$$

The local isometry classes of the metric (5.11he) *are parameterized by three arbitrary functions of two variables modulo two arbitrary functions of one variable.*

Proof. According to the second part of Proposition 4.9 we have $h \neq 0$ and according to Proposition 5.1 we can assume $H(w, x) = x$ and $h = 1$. Then (5.11he) follows from (5.1he) and (1.8). By (5.5he), the condition $\alpha \neq 0$ implies $\mathcal{D} \neq 0$ or $\mathcal{E} \neq 0$. Further we have $q\,\mathcal{D} - p\,\mathcal{E} \neq 0$, which follows from (5.4) and a condition $pu - qt \neq 0$ for the non-orthogonal foliation.

In the following we will assume that $\mathcal{E} \neq 0$. (The case $\mathcal{D} \neq 0$ is completely analogous). We express (5.4) in the form $t = u\,\mathcal{D}/\mathcal{E}$ and substitute into (5.7). Here $v = \mathcal{E}/u$, which follows from (5.5he) and (5.6). We obtain from (5.7) that

$$\text{(5.15)} \quad (u^2)'_x - Pu^2 = Q,$$

where P and Q are functions given by (5.14a). Finally, we can solve (5.15) by the standard method of "variation of constants". This proves the first part of Theorem 5.5.

We now prove the statement about the local isometry classes. Let (M, g) and $(\bar{M}, \bar{g})$ be two spaces with the metrics of the form (5.11he) and let $\Phi : M \longrightarrow \bar{M}$ be an isometry. We shall denote the corresponding functions and forms for $(\bar{M}, \bar{g})$ by bars and we make the usual conventions (see the end of Section 3). Because $b = 0$ and $h \neq 0$, $(3.10)_2$ implies $c \neq 0$ and we can use Proposition 3.3. Let us assume, for simplicity, $\varepsilon_2 = \varepsilon_3 = 1$ (for the other signs, the argument is similar). We get first $\bar{\omega}^1 = \varepsilon\omega^1$ and hence there is a function $\varphi = \varphi(w)$ of the variable w only such that

$$\text{(5.16)} \quad \bar{w} = \varphi, \quad \mathrm{d}\bar{w} = \varphi'\mathrm{d}w.$$

The equation $\bar{\omega}^3 = \omega^3$ means $\mathrm{d}(\bar{y}-y) = (x-\bar{x}\varphi')\mathrm{d}w$, *i.e,* there is a function $\psi = \psi(w)$ of the variable w only such that

$$(5.17)\quad \bar{y} = y + \psi, \qquad \bar{x} = \frac{x-\psi'}{\varphi'}.$$

Finally, we substitute from (5.16) and (5.17) into the equation $\bar{\omega}^2 = \omega^2$. Comparing the coefficients of $\mathrm{d}x$ and $\mathrm{d}w$, respectively, we obtain, in the hyperbolic case,

$$(5.18\mathrm{h})\quad \begin{cases} \bar{p} = \varphi'[p\cosh(\lambda\psi) - q\sinh(\lambda\psi)], \\ \bar{q} = \varphi'[q\cosh(\lambda\psi) - p\sinh(\lambda\psi)], \\ \bar{r} = \dfrac{r - p\bar{x}'_w\varphi'}{\varphi'}\cosh(\lambda\psi) - \dfrac{s - q\bar{x}'_w\varphi'}{\varphi'}\sinh(\lambda\psi), \\ \bar{s} = \dfrac{s - q\bar{x}'_w\varphi'}{\varphi'}\cosh(\lambda\psi) - \dfrac{r - p\bar{x}'_w\varphi'}{\varphi'}\sinh(\lambda\psi), \end{cases}$$

where $\bar{x}'_w$ can be calculated from $(5.17)_2$. In the elliptic case we obtain analogous formulas (5.18e) in which "cosh" and " sinh " are replaced by "cos" and "sin" respectively. Further, we always have $\lambda(\bar{q}\bar{r} - \bar{p}\bar{s}) = \lambda(qr - ps) = 1$. The formulas (5.16), (5.17) and (5.18he) show that the functions $\bar{w}$, $\bar{x}$, $\bar{y}$, $\bar{p}$, $\bar{q}$, $\bar{r}$, $\bar{s}$, $\bar{t}$ and $\bar{u}$ can be expressed through w, x, y, p, q, r, s, t, u and the two arbitrary functions φ and ψ of one variable. Thus each local isometry class depends on two arbitrary functions of one variable.

This completes the proof. ∎

Remark. The isometry part of Theorem 5.5 (and of some other theorems which follow) can be stated more precisely using the concept of germs (*cf.* [18]).

Remark. In Theorem 11.38 of [3] the formulas (11.114) contain a misprint. The coefficients 2 should be in the numerators (not in the denominators) of the formulas for the both functions P and Q.

Theorem 5.6. *The metric of a three-dimensional non-orthogonally foliated singular space of type* (H) *is locally determined, in the hyperbolic case, by an orthonormal coframe*

$$(5.19\mathrm{h})\quad \begin{cases} \omega^1 = [t\cosh(\lambda y) + u\sinh(\lambda y)]\mathrm{d}w, \\ \omega^2 = [p\cosh(\lambda y) + (\varphi - p)\sinh(\lambda y)]\mathrm{d}x + r[\cosh(\lambda y) - \sinh(\lambda y)]\mathrm{d}w, \\ \omega^3 = \mathrm{d}y + H\mathrm{d}w, \end{cases}$$

where $\varphi = \varphi(w,x)$ is an arbitrary nonzero function of the variables w and x such that $(\ln|\varphi|)''_{wx} \neq 0$ *on an open set, and*

$$p = \frac{1}{2}\varphi + \frac{(\ln|\varphi|)''_{xx}}{\lambda^2\varphi} - \frac{[(\ln|\varphi|)'_x]^2}{2\lambda^2\varphi} + \frac{\psi}{\varphi},$$

$$r = \frac{(\ln|\varphi|)''_{wx}}{\lambda^2\varphi}, \quad H = \frac{(\ln|\varphi|)'_w}{\lambda},$$

where $\psi = \psi(x)$ is an arbitrary function of the variable x. Further, t can be chosen as an arbitrary function of the variables w and x, and $u = \int(\varphi/p - 1)t'_x\,\mathrm{d}x$.

Proof. The singularity condition $\alpha = 0$ implies $\mathcal{D} = \mathcal{E} = 0$. Our necessary conditions are

$$(5.20\text{h}) \quad \begin{cases} p'_w - r'_x - \lambda q H = 0, \\ q'_w - s'_x - \lambda p H = 0, \\ H'_x + \lambda(ps - qr) = 0, \\ p u'_x - q t'_x = 0. \end{cases}$$

We write $\omega^2 = A\,\mathrm{d}x + C\,\mathrm{d}w$ in the form

$$\omega^2 = \frac{1}{2}[(p+q)\mathrm{e}^{\lambda y} + (p-q)\mathrm{e}^{-\lambda y}]\mathrm{d}x + \frac{1}{2}[(r+s)\mathrm{e}^{\lambda y} + (r-s)\mathrm{e}^{-\lambda y}]\mathrm{d}w.$$

We see that either $p+q \neq 0$ or $p-q \neq 0$ on some open set because $A \neq 0$. Assume that $p+q \neq 0$ (the other case is analogous). Taking a new variable $\bar{x} = \bar{x}(w,x)$ (instead of x) as a potential function of the Pfaffian equation $(p+q)\mathrm{d}x + (r+s)\mathrm{d}w = 0$, we can put $r+s = 0$. Then $(5.20\text{h})_{1-3}$ are rewritten, in the standard notation, as

$$(5.21\text{h}) \quad \begin{cases} -r'_x + p'_w - \lambda q H = 0, \\ r'_x + q'_w - \lambda p H = 0, \\ H'_x - \lambda(p+q) r = 0. \end{cases}$$

Here we choose $\varphi = p+q$ as an arbitrary nonzero function such that $(\ln|\varphi|)''_{wx} \neq 0$. By $(5.21\text{h})_{1,2}$ we get first $H = (\ln|\varphi|)'_w/\lambda$ and hence $h = H'_x \neq 0$ (hyperbolicity condition). Further, by $(5.21\text{h})_3$, $r = (\ln|\varphi|)''_{wx}/(\lambda^2\varphi)$. Now $(5.21\text{h})_1$ means $p'_w + (\ln|\varphi|)'_w p = \varphi'_w + r'_x$, which can be solved by the standard method of "variation of constants". Finally the function t is chosen as arbitrary and u is calculated from $(5.20\text{h})_4$. ∎

Theorem 5.7. *The metrics of three-dimensional non-orthogonally foliated singular spaces of type* (H) *belong, in the elliptic case, to one of the following two classes*:
Class I. *The metric is locally determined by an orthonormal coframe*

$$(5.19\text{e-}1) \quad \begin{cases} \omega^1 = [t\cos(\lambda y) + u\sin(\lambda y)]\mathrm{d}w, \\ \omega^2 = [p\cos(\lambda y) + q\sin(\lambda y)]\mathrm{d}x + s\sin(\lambda y)\mathrm{d}w, \\ \omega^3 = \mathrm{d}y + H\mathrm{d}w, \end{cases}$$

where H is an arbitrary function of the variables w and x such that $HH'_x \neq 0$, p is any non-trivial solution of the second order partial differential equation

$$p''_{ww} - \frac{H'_w}{H}p'_w - \frac{HH'_x}{p^2}p'_x + \lambda^2 H^2 p + \frac{HH''_{xx}}{p} = 0,$$

$q = p'_w/(\lambda H)$ and $s = -H'_x/(\lambda p)$. Further, t can be chosen as an arbitrary function of the variables w and x, and $u = \int p'_w t'_x/(\lambda p H)\,\mathrm{d}x$.

Class II. *The metric is locally determined by an orthonormal coframe*

$$\text{(5.19e-2)}\quad \begin{cases} \omega^1 = [\cos(\lambda y) + u\sin(\lambda y)]\mathrm{d}w, \\ \omega^2 = \sin(\lambda y)\mathrm{d}x + [\psi\cos(\lambda x) - \varphi\sin(\lambda x)]\cos(\lambda y)\mathrm{d}w, \\ \omega^3 = \mathrm{d}y + [\varphi\cos(\lambda x) + \psi\sin(\lambda x)]\mathrm{d}w, \end{cases}$$

where u is an arbitrary function of the variables w and x, and $\varphi = \varphi(w)$ and $\psi = \psi(w)$ are arbitrary functions of the variable w such that $\varphi^2 + \psi^2 \neq 0$.

Proof. The singularity condition $\alpha = 0$ implies again $\mathcal{D} = \mathcal{E} = 0$. Our necessary conditions are now

$$\text{(5.20e)}\quad \begin{cases} p'_w - r'_x - \lambda qH = 0, \\ q'_w - s'_x + \lambda pH = 0, \\ H'_x + \lambda(ps - qr) = 0, \\ pu'_x - qt'_x = 0. \end{cases}$$

Taking a new variable $\bar{x} = \bar{x}(w, x)$ as a potential function of the Pfaffian equation $p\,\mathrm{d}x + r\,\mathrm{d}w = 0$, we can put $r = 0$. We distinguish two cases: $p \neq 0$ and $p = 0$.

Case I. The function p is nonzero on some open set. Let H be an arbitrary function of the variables w and x such that $HH'_x \neq 0$. Then, from $(5.20\text{e})_1$ with $r = 0$ and $(5.20\text{e})_3$, we obtain

$$\text{(5.21e)}\quad q = \frac{p'_w}{\lambda H}, \quad s = -\frac{H'_x}{\lambda p},$$

respectively. Substituting (5.21e) into $(5.20\text{e})_2$, we obtain

$$\text{(5.22e)}\quad p''_{ww} - \frac{H'_w}{H}p'_w - \frac{HH'_x}{p^2}p'_x + \lambda^2 H^2 p + \frac{HH''_{xx}}{p} = 0,$$

which is a second order partial differential equation for p. According to the Cauchy-Kowalewski Theorem, the general solution of (5.22e) depends on two arbitrary functions of one variable. Now the functions q and s are calculated by (5.21e). Finally let t be an arbitrary function of the variables w and x. Then we obtain u from $(5.20\text{e})_4$ by integration.

Case II. The function p is identically zero on some open set. In this case we have $q \neq 0$ because $A \neq 0$. The necessary conditions (5.20e) reduce to

$$\text{(5.23e)}\quad \begin{cases} r'_x + \lambda qH = 0, \\ q'_w - s'_x = 0, \\ H'_x - \lambda qr = 0, \\ t'_x = 0. \end{cases}$$

The equation $(5.23\text{e})_2$ means that $q\,\mathrm{d}x + s\,\mathrm{d}w$ is an exact differential form. Let $\bar{x} = \bar{x}(w, x)$ be a new variable such that $\mathrm{d}\bar{x} = q\,\mathrm{d}x + s\,\mathrm{d}w$. Further, let $\bar{w} = \bar{w}(w)$ be a new variable satisfying

$d\bar{w} = t\,dw$. Then, taking the new system of coordinates $(\bar{w}, \bar{x}, y)$ (which we denote again as (w, x, y)) we can make $q = t = 1$ and $s = 0$ in (5.1he). The equations $(5.23e)_{1,3}$ read $H''_{xx} + \lambda^2 H = 0$, hence $H = \varphi\cos(\lambda x) + \psi\sin(\lambda x)$, and from $(5.23e)_3$ we get $r = \psi\cos(\lambda x) - \varphi\sin(\lambda x)$, where $\varphi = \varphi(w)$ and $\psi = \psi(w)$ are functions of the variable w such that $\varphi^2 + \psi^2 \neq 0$. ∎

Remark. In Class I, if we choose the basic functions H and p as depending only on x, we can obtain a family of metric in an explicit form.

5.2 The orthogonally foliated spaces of type (H)

For the metrics of type (H) it may also happen $h = 0$ everywhere (and thus $H = 0$ due to Proposition 5.1), which means, according to Proposition 4.9, that the asymptotic foliations are mutually orthogonal. This case was excluded in the previous section. As we see from (5.9), one of the asymptotic distributions is given by the equation $\omega^1 = 0$. Consequently, the second must be given by the equation $\omega^2 = 0$. Thus, the equation (5.10) implies that $(C/A)'_y = 0$. Hence, there exists a function $\mu = \mu(w, x)$ of the variables w and x such that $C = \mu A$, and hence, in the hyperbolic case,

$$\begin{cases} \omega^1 = [t\cosh(\lambda y) + u\sinh(\lambda y)]dw, \\ \omega^2 = [p\cosh(\lambda y) + q\sinh(\lambda y)](dx + \mu dw), \\ \omega^3 = dy \end{cases}$$

and, in the elliptic case,

$$\begin{cases} \omega^1 = [t\cos(\lambda y) + u\sin(\lambda y)]dw, \\ \omega^2 = [p\cos(\lambda y) + q\sin(\lambda y)](dx + \mu dw), \\ \omega^3 = dy. \end{cases}$$

Introducing the new variable $\bar{x}$ instead of x, where $\bar{x} = \bar{x}(w, x)$ is a potential function of the Pfaffian equation $dx + \mu\,dw = 0$, we get $C = 0$.

The conditions (5.4) and (5.7) now read (in the standard notation)

$$(5.24)\quad \begin{cases} up'_w - tq'_w = 0, \\ pu'_x - qt'_x = 0. \end{cases}$$

Moreover, we must have $pu - qt \neq 0$, otherwise the metrics would be not of type (H). We put $\varphi = pu - qt$ and rewrite (5.24) in the equivalent form

$$(5.25)\quad \begin{cases} up'_w - tq'_w = 0, \\ up'_x - tq'_x = \varphi'_x, \\ pu - qt = \varphi. \end{cases}$$

Again we distinguish two geometric situations: the generic case and the singular case.

We first treat the *generic* case where the asymptotic distribution given by the equation $\omega^1 = 0$ is not totally geodesic. We remember that this requirement is equivalent to $\alpha \neq 0$.

Theorem 5.8. *The metric of a three-dimensional orthogonally foliated generic space of type* (H) *is locally determined, in the hyperbolic case, by an orthonormal coframe*

$$(5.26h)\quad \begin{cases} \omega^1 = [t\cosh(\lambda y) + u\sinh(\lambda y)]\mathrm{d}w, \\ \omega^2 = [p\cosh(\lambda y) + q\sinh(\lambda y)]\mathrm{d}x, \\ \omega^3 = \mathrm{d}y \end{cases}$$

and, in the elliptic case, by an orthonormal coframe

$$(5.26e)\quad \begin{cases} \omega^1 = [t\cos(\lambda y) + u\sin(\lambda y)]\mathrm{d}w, \\ \omega^2 = [p\cos(\lambda y) + q\sin(\lambda y)]\mathrm{d}x, \\ \omega^3 = \mathrm{d}y, \end{cases}$$

where p and q are arbitrary functions of the variables w and x satisfying $pq'_w - qp'_w \neq 0$, and t and u are calculated from p and q by

$$(5.27)\quad t = \frac{\varphi p'_w}{pq'_w - qp'_w}, \quad u = \frac{\varphi q'_w}{pq'_w - qp'_w},$$

where

$$(5.28)\quad |\varphi| = \exp\left(\int \frac{p'_x q'_w - p'_w q'_x}{pq'_w - p'_w q}\,\mathrm{d}x\right).$$

The local isometry classes are parameterized by two arbitrary functions of two variables modulo two arbitrary functions of one variable and a real parameter.

Proof. The requirement $\alpha \neq 0$ is equivalent to $pq'_w - qp'_w \neq 0$. Indeed, if $\alpha \neq 0$, then, by (5.3) and (5.5he), $\mathcal{D} = p'_w$ or $\mathcal{E} = q'_w$ is different from zero. Together with the condition $pu - qt \neq 0$ and $(5.25)_1$, this implies $pq'_w - qp'_w \neq 0$. Conversely, if $pq'_w - qp'_w \neq 0$, then at least one of the functions $\mathcal{D}$ and $\mathcal{E}$ is nonzero, and hence $\alpha \neq 0$.

From $(5.25)_1$ and $(5.25)_3$ we can then express t and u by the Cramer's rule. We obtain (5.27). Substituting from here in $(5.25)_2$, we have a differential equation for the function φ:

$$\varphi'_x = \frac{p'_x q'_w - p'_w q'_x}{pq'_w - qp'_w}\varphi,$$

which can be solved explicitly in the form (5.28).

The assertion about the local isometry classes can be proved exactly in the same way as we did for non-orthogonally foliated spaces of type (H). ∎

It remains the *singular* case where the asymptotic distribution given by the equation $\omega^1 = 0$ is totally geodesic.

Theorem 5.9. *The metric of three-dimensional orthogonally foliated singular spaces of type* (H) *belong to the following two classes*:

Class I. *The orthonormal coframe is given, in the hyperbolic case, by*

$$\text{(5.29h)}\quad \begin{cases} \omega^1 = [t\cosh(\lambda y) + u\sinh(\lambda y)]\mathrm{d}w, \\ \omega^2 = [\cosh(\lambda y) + q\sinh(\lambda y)]\mathrm{d}x, \\ \omega^3 = \mathrm{d}y \end{cases}$$

and, in the elliptic case, by

$$\text{(5.29e)}\quad \begin{cases} \omega^1 = [t\cos(\lambda y) + u\sin(\lambda y)]\mathrm{d}w, \\ \omega^2 = [\cos(\lambda y) + q\sin(\lambda y)]\mathrm{d}x, \\ \omega^3 = \mathrm{d}y, \end{cases}$$

where q is an arbitrary function of the variable x, t is an arbitrary function of the variables w and x, and the function u has the form $u = \int qt'_x\,\mathrm{d}x$.

Class II. *The orthonormal coframe is given, in the hyperbolic case, by*

$$\text{(5.30h)}\quad \begin{cases} \omega^1 = [\cosh(\lambda y) + u\sinh(\lambda y)]\mathrm{d}w, \\ \omega^2 = \sinh(\lambda y)\mathrm{d}x, \\ \omega^3 = \mathrm{d}y \end{cases}$$

and, in the elliptic case, by

$$\text{(5.30e)}\quad \begin{cases} \omega^1 = [\cos(\lambda y) + u\sin(\lambda y)]\mathrm{d}w, \\ \omega^2 = \sin(\lambda y)\mathrm{d}x, \\ \omega^3 = \mathrm{d}y, \end{cases}$$

where the function u is an arbitrary function of the variables w and x.

Proof. We have $p'_w = q'_w = 0$. Hence $(5.24)_{1,2}$ reduce to only one equation

$$pu'_x - qt'_x = 0.$$

If $p = p(x) \neq 0$, then we can make $p = 1$ in (5.1he). Indeed, we choose a new variable $\bar{x} = \bar{x}(x)$ satisfying $\mathrm{d}\bar{x} = p\,\mathrm{d}x$. Hence we have $u = \int qt'_x\,\mathrm{d}x$ for arbitrary functions $q = q(x)$ and $t = t(w, x)$.

If $p = 0$, then necessarily $q = q(x) \neq 0$. Here, we can make $q = 1$ in (5.1he). Further, choosing a new variable $\bar{w} = \bar{w}(w)$ satisfying $\mathrm{d}\bar{w} = t\,\mathrm{d}w$, we can make $t = 1$ in (5.1he). The function u is an arbitrary function of the variables w and x. ∎

Remark. (1) In all cases we shall ensure, by choosing proper initial conditions, that $\varphi = pu - qt \neq 0$.

(2) The asymptotic distribution given by the equation $\omega^2 = 0$ is totally geodesic if and only if $t'_x = u'_x = 0$ in the above expression.

5.3 The spaces of type (P)

The only asymptotic distribution on any space of type (P) is defined by the equation $\omega^1 = 0$.

According to Corollary 4.5 and Proposition 5.1, all we have to assume is $H = x$, $h = 1$ and the parabolicity condition $f = \xi A$ with some nonzero function $\xi = \xi(w, x)$ of the variables w and x. Thus the only algebraic relations for the basic functions are

$$(5.31)\quad \lambda(qr - ps) = 1$$

and

$$(5.32)\quad t = \xi p, \quad u = \xi q.$$

Now, we put

$$(5.33)\quad \begin{cases} D = \lambda(p'_x q - p q'_x), \\ E = \lambda[p q'_w - p'_w q + \lambda(\epsilon p^2 + q^2)x]. \end{cases}$$

(Here E is different for the hyperbolic case and for elliptic case). Taking use of (5.7) and (5.32), we can rewrite (5.6) as

$$(5.34)\quad \xi D = v.$$

We can also rewrite (5.4) as

$$(5.35)\quad q\,\mathcal{D} - p\,\mathcal{E} = 0,$$

which means

$$\lambda(p s'_x - q r'_x) = E$$

or, due to (5.31),

$$(5.36)\quad \lambda(q'_x r - p'_x s) = E.$$

Theorem 5.10. *The metric of a three-dimensional foliated generic space of type* (P) *is given, in the hyperbolic case, by*

$$(5.37\text{h})\quad \begin{cases} \omega^1 = \xi[p\cosh(\lambda y) + q\sinh(\lambda y)]\mathrm{d}w, \\ \omega^2 = [p\cosh(\lambda y) + q\sinh(\lambda y)]\mathrm{d}x + [r\cosh(\lambda y) + s\sinh(\lambda y)]\mathrm{d}w, \\ \omega^3 = \mathrm{d}y + x\,\mathrm{d}w \end{cases}$$

and, in the elliptic case, by

$$(5.37\text{e})\quad \begin{cases} \omega^1 = \xi[p\cos(\lambda y) + q\sin(\lambda y)]\mathrm{d}w, \\ \omega^2 = [p\cos(\lambda y) + q\sin(\lambda y)]\mathrm{d}x + [r\cos(\lambda y) + s\sin(\lambda y)]\mathrm{d}w, \\ \omega^3 = \mathrm{d}y + x\,\mathrm{d}w, \end{cases}$$

where p and q are arbitrary functions of the variables w and x, $qp'_x - pq'_x \neq 0$*, and*

$$\text{(5.38)}\quad \begin{cases} r = \dfrac{p'_x - pE}{D}, \qquad s = \dfrac{q'_x - qE}{D}, \\ \xi = \left[\dfrac{p'_w - r'_x - \lambda q x}{pD}\right]^{1/2} = \left[\dfrac{q'_w - s'_x - \epsilon\lambda p x}{qD}\right]^{1/2}. \end{cases}$$

The local isometry classes are parameterized by two arbitrary functions of two variables modulo two arbitrary functions of one variable.

Proof. According to (5.6), (5.34) and (5.33), $\alpha \neq 0$ holds if and only if $D \neq 0$ or, equivalently, $qp'_x - pq'_x \neq 0$.

The relations (5.31) and (5.36) form a system of linear algebraic equations for r and s with the coefficients depending on p and q only. Hence we get the expression $(5.38)_{1,2}$ by the Cramer's rule. Now, the equations (5.5he), (5.6) and (5.34) imply that $\xi D = \mathcal{D}/(\xi p) = \mathcal{E}/(\xi q)$, from which we can determine ξ in the form $(5.38)_3$. Because either $p \neq 0$ or $q \neq 0$, at least one expression for ξ is correct. Let us notice that the functions p and q only have to satisfy certain differential inequalities and thus p and q can still be considered as arbitrary functions of two variables.

The assertion about the local isometry classes can be proved exactly in the same way as we did for the type (H). ∎

Theorem 5.11. *The metric of a three-dimensional foliated singular space of type* (P) *is given, in the hyperbolic case, by*

$$\text{(5.39h-1)}\quad \begin{cases} \omega^1 = \xi p \cosh(\lambda y)\mathrm{d}w, \\ \omega^2 = p\cosh(\lambda y)\mathrm{d}x + [r\cosh(\lambda y) + s\sinh(\lambda y)]\mathrm{d}w, \\ \omega^3 = \mathrm{d}y + x\,\mathrm{d}w \end{cases}$$

with

$$\text{(5.40h-1)}\quad \begin{cases} p = \dfrac{1}{\sqrt{\psi + \lambda^2 x^2}}, \\ r = -\dfrac{\psi' x}{2\psi\sqrt{\psi + \lambda^2 x^2}} + \varphi, \quad s = -\lambda\sqrt{\psi + \lambda^2 x^2} \end{cases}$$

or

$$\text{(5.39h-2)}\quad \begin{cases} \omega^1 = \xi q \sinh(\lambda y)\mathrm{d}w, \\ \omega^2 = q\sinh(\lambda y)\mathrm{d}x + [r\cosh(\lambda y) + s\sinh(\lambda y)]\mathrm{d}w, \\ \omega^3 = \mathrm{d}y + x\,\mathrm{d}w \end{cases}$$

with

$$\text{(5.40h-2)}\quad \begin{cases} q = \dfrac{1}{\sqrt{\psi - \lambda^2 x^2}}, \\ r = \lambda\sqrt{\psi - \lambda^2 x^2}, \quad s = -\dfrac{\psi' x}{2\psi\sqrt{\psi - \lambda^2 x^2}} + \varphi \end{cases}$$

and, in the elliptic case, by

$$(5.39e)\quad \begin{cases} \omega^1 = \xi p \cos(\lambda y) \mathrm{d}w, \\ \omega^2 = p\cos(\lambda y)\mathrm{d}x + [r\cos(\lambda y) + s\sin(\lambda y)]\mathrm{d}w, \\ \omega^3 = \mathrm{d}y + x\,\mathrm{d}w, \end{cases}$$

with

$$(5.40e)\quad \begin{cases} p = \dfrac{1}{\sqrt{\psi - \lambda^2 x^2}}, \\ r = -\dfrac{\psi' x}{2\psi\sqrt{\psi - \lambda^2 x^2}} + \varphi, \quad s = -\lambda\sqrt{\psi - \lambda^2 x^2}, \end{cases}$$

where $\xi = \xi(w, x)$ is an arbitrary nonzero function of the variables w and x, and $\varphi = \varphi(w)$ and $\psi = \psi(w)$ are arbitrary functions of the variable w.

Proof. The requirement $\alpha = 0$ is equivalent to $qp'_x - pq'_x = 0$.

In case that $p \neq 0$ we see from $(q/p)'_x = 0$ that $q = \kappa p$ for some function $\kappa = \kappa(w)$ of the variable w. In the hyperbolic case, taking a function $\theta = \theta(w)$ of the variable w such that $\tanh\theta = \kappa$, we consider new variables $\bar{x} = x - \theta'/\lambda$ and $\bar{y} = y + \theta/\lambda$. In the coordinates $(w, \bar{x}, \bar{y})$, the orthonormal coframe (5.37h) is expressed as

$$\begin{cases} \omega^1 = \xi p\sqrt{1-\kappa^2}\cosh(\lambda\bar{y})\mathrm{d}w, \\ \omega^2 = p\sqrt{1-\kappa^2}\cosh(\lambda\bar{y})\mathrm{d}\bar{x} + [\bar{r}\cosh(\lambda\bar{y}) + \bar{s}\sinh(\lambda\bar{y})]\mathrm{d}w, \\ \omega^3 = \mathrm{d}\bar{y} + \bar{x}\,\mathrm{d}w \end{cases}$$

with $\bar{r} = r\cosh\theta - s\sinh\theta - p\sqrt{1-\kappa^2}\,\theta''/\lambda$ and $\bar{s} = s\cosh\theta - r\sinh\theta$. In the elliptic case, taking a function $\theta = \theta(w) \in (-\pi/2, \pi/2)$ of the variable w such that $\tan\theta = \kappa$, we consider new variables $\bar{x} = x + \theta'/\lambda$ and $\bar{y} = y - \theta/\lambda$. In the coordinates $(w, \bar{x}, \bar{y})$, the orthonormal coframe (5.37e) is expressed as

$$\begin{cases} \omega^1 = \xi p\sqrt{1+\kappa^2}\cos(\lambda\bar{y})\mathrm{d}w, \\ \omega^2 = p\sqrt{1+\kappa^2}\cos(\lambda\bar{y})\mathrm{d}\bar{x} + [\bar{r}\cos(\lambda\bar{y}) + \bar{s}\sin(\lambda\bar{y})]\mathrm{d}w, \\ \omega^3 = \mathrm{d}\bar{y} + \bar{x}\,\mathrm{d}w \end{cases}$$

with $\bar{r} = r\cos\theta - s\sin\theta + p\sqrt{1+\kappa^2}\,\theta''/\lambda$ and $\bar{s} = s\cos\theta - r\sin\theta$. Thus we can put $q = 0$ in (5.37he) and we obtain (5.39h-1) and (5.39e), respectively. Then the equations (5.36), $\mathcal{D} = 0$ and (5.31) reduce to

$$(5.41)\quad \begin{cases} sp'_x = -\epsilon\lambda p^2 x, \\ r'_x = p'_w, \\ \lambda p s = -1. \end{cases}$$

The equations $(5.41)_1$ and $(5.41)_3$ imply $p'_x = \epsilon\lambda^2 p^3 x$, hence we obtain (5.40h-1) and (5.40e) with arbitrary functions $\varphi = \varphi(w)$ and $\psi = \psi(w)$ of the variable w.

If $p = 0$ identically, we have $q \neq 0$ because $A \neq 0$. The equations (5.36), $\mathcal{E} = 0$ and (5.31) reduce to

$$(5.42)\quad \begin{cases} rq'_x = \lambda q^2 x, \\ s'_x = q'_w, \\ \lambda qr = 1. \end{cases}$$

The equations $(5.42)_1$ and $(5.42)_3$ imply $q'_x = \lambda^2 p^3 x$, hence we obtain (5.40h-2) with arbitrary functions $\varphi = \varphi(w)$ and $\psi = \psi(w)$ of the variable w. We notice that we obtain a new class of metrics, (5.39h-2), in the hyperbolic case, but not in the elliptic case. Indeed, in the elliptic case, taking a new variable $\bar{y} = y + \pi/(2\lambda)$, we can reduce the second case $p = 0$ to the first one. ∎

5.4 The spaces of type (Pℓ)

We are now left with the type (Pℓ) in which there are no singular solutions.

Theorem 5.12. *The metric of a three-dimensional foliated space of type* (Pℓ) *is locally determined, in the hyperbolic case, by an orthonormal coframe*

$$(5.43\text{h-1})\quad \begin{cases} \omega^1 = \xi \sinh(\lambda y)\mathrm{d}w, \\ \omega^2 = \sinh(\lambda y)\mathrm{d}x, \\ \omega^3 = \mathrm{d}y \end{cases}$$

or

$$(5.43\text{h-2})\quad \begin{cases} \omega^1 = \xi \cosh(\lambda y)\mathrm{d}w, \\ \omega^2 = \cosh(\lambda y)\mathrm{d}x, \\ \omega^3 = \mathrm{d}y \end{cases}$$

and, in the elliptic case, by an orthonormal coframe

$$(5.43\text{e})\quad \begin{cases} \omega^1 = \xi \sin(\lambda y)\mathrm{d}w, \\ \omega^2 = \sin(\lambda y)\mathrm{d}x, \\ \omega^3 = \mathrm{d}y, \end{cases}$$

where $\xi = \xi(w, x)$ is a nonzero function of the variables w and x. The local isometry classes are parameterized by the function ξ modulo two arbitrary functions of one variable.

Proof. According to Corollary 4.5, we have $f = \xi A$, $C = \zeta A$ and $a_0 = 0$, and we get $h = 0$. We can write due to (5.1he)

$$(5.44)\quad t = \xi p, \quad u = \xi q$$

and

(5.45) $r = \zeta p, \quad s = \zeta q.$

If we substitute (5.44) and $h = 0$ into the equation (5.7), we obtain

(5.46) $p'_x q - p q'_x = 0.$

Substituting in (5.4) from (5.3), and using (5.45) and (5.46), we get

(5.47) $p'_w q - p q'_w = 0.$

In case that $q \neq 0$, (5.46) and (5.47) imply that p/q is a constant, say a. We can express the orthonormal coframe (1.8) in the form, in the hyperbolic case,

$$\begin{cases} \omega^1 = \xi q[a\cosh(\lambda y) + \sinh(\lambda y)]\mathrm{d}w, \\ \omega^2 = q[a\cosh(\lambda y) + \sinh(\lambda y)](\mathrm{d}x + \zeta \mathrm{d}w), \\ \omega^3 = \mathrm{d}y \end{cases}$$

and, in the elliptic case,

$$\begin{cases} \omega^1 = \xi q[a\cos(\lambda y) + \sin(\lambda y)]\mathrm{d}w, \\ \omega^2 = q[a\cos(\lambda y) + \sin(\lambda y)](\mathrm{d}x + \zeta \mathrm{d}w), \\ \omega^3 = \mathrm{d}y. \end{cases}$$

Taking a constant $\omega \in \mathbb{R}$ such that $\tanh\omega = a$ in the hyperbolic case, and $\omega \in (-\pi/2, \pi/2)$ such that $\tan\omega = a$ in the elliptic case, we obtain, in the hyperbolic case,

$$a\cosh(\lambda y) + \sinh(\lambda y) = \sqrt{1-a^2}\,\sinh(\lambda y + \omega)$$

and, in the elliptic case,

$$a\cos(\lambda y) + \sin(\lambda y) = \sqrt{1+a^2}\,\sin(\lambda y + \omega).$$

Hence, substituting the new variable $\bar{y} = y + \omega/\lambda$, we obtain, in the hyperbolic case,

$$\text{(5.48h)} \quad \begin{cases} \omega^1 = \xi q\sqrt{1-a^2}\,\sinh(\lambda\bar{y})\mathrm{d}w, \\ \omega^2 = q\sqrt{1-a^2}\,\sinh(\lambda\bar{y})(\mathrm{d}x + \zeta \mathrm{d}w), \\ \omega^3 = \mathrm{d}\bar{y} \end{cases}$$

and, in the elliptic case,

$$\text{(5.48e)} \quad \begin{cases} \omega^1 = \xi q\sqrt{1+a^2}\,\sin(\lambda\bar{y})\mathrm{d}w, \\ \omega^2 = q\sqrt{1+a^2}\,\sin(\lambda\bar{y})(\mathrm{d}x + \zeta \mathrm{d}w), \\ \omega^3 = \mathrm{d}\bar{y}. \end{cases}$$

Let us introduce the new variable $\bar{x} = \bar{x}(w, x)$ as a potential function of the Pfaffian equation $\mathrm{d}x + \zeta \mathrm{d}w = 0$. Then we can write (5.48he) in the form, in the hyperbolic case,

$$\omega^1 = u \sinh(\lambda\bar{y})\mathrm{d}w, \quad \omega^2 = \bar{q} \sinh(\lambda\bar{y})\mathrm{d}\bar{x}, \quad \omega^3 = \mathrm{d}\bar{y}$$

and, in the elliptic case,

$$\omega^1 = u \sin(\lambda\bar{y})\mathrm{d}w, \quad \omega^2 = \bar{q} \sin(\lambda\bar{y})\mathrm{d}\bar{x}, \quad \omega^3 = \mathrm{d}\bar{y}.$$

Now, solving a first order linear partial differential equation, we can find a function $\theta = \theta(w, \bar{x})$ such that $(\cos\theta)u\mathrm{d}w + (\sin\theta)\bar{q}\mathrm{d}\bar{x}$ is (locally) a total differential, say $\mathrm{d}X$. Using a new orthonormal coframe $\{\bar{\omega}^1, \bar{\omega}^2, \omega^3\}$, where $\bar{\omega}^1 = (\sin\theta)\omega^1 - (\cos\theta)\omega^2$, $\bar{\omega}^2 = (\cos\theta)\omega^1 + (\sin\theta)\omega^2$, and denoting by W a potential function of the Pfaffian equation $(\sin\theta)u\mathrm{d}w - (\cos\theta)\bar{q}\mathrm{d}\bar{x} = 0$, we obtain, in the hyperbolic case,

$$\bar{\omega}^1 = \bar{\xi} \sinh(\lambda\bar{y})\mathrm{d}W, \quad \bar{\omega}^2 = \sinh(\lambda\bar{y})\mathrm{d}X$$

and, in the elliptic case,

$$\bar{\omega}^1 = \bar{\xi} \sin(\lambda\bar{y})\mathrm{d}W, \quad \bar{\omega}^2 = \sin(\lambda\bar{y})\mathrm{d}X,$$

which gives (5.43h-1) or (5.43e) up to notation.

If now $q = 0$ identically, we obtain, in the hyperbolic case,

$$\omega^1 = \xi p \cosh(\lambda y)\mathrm{d}w, \quad \omega^2 = p \cosh(\lambda y)(\mathrm{d}x + \zeta\mathrm{d}w), \quad \omega^3 = \mathrm{d}y$$

which can be reduced to the form (5.43h-2). In the elliptic case we get

$$\omega^1 = \xi p \cos(\lambda y)\mathrm{d}w, \quad \omega^2 = p \cos(\lambda y)(\mathrm{d}x + \zeta\mathrm{d}w), \quad \omega^3 = \mathrm{d}y$$

and this is again reduced to (5.43e) after the substitution $\bar{y} = y + \pi/(2\lambda)$.

Next, we look at the local isometry classes of metrics of the form (5.43h-1), (5.43h-2) or (5.43e). Because of the geometric meaning of the foliation variable $\bar{y}$ and the specific form of (5.43h-1), (5.43h-2) or (5.43e), it follows easily that for two isometric metrics of the form (5.43h-1), (5.43h-2) or (5.43e) the foliation variable will be the same, possibly up to sign. But then we see that the problem of characterizing the local isometry classes of the metrics (5.43h-1), (5.43h-2) or (5.43e) is (locally) the same as to classify the surfaces in $\mathbb{E}^3$ up to an isometry. This problem was solved (in the analytic case) by E. Cartan: the set of all surfaces in $\mathbb{E}^3$ which are (locally) isometric to a fixed generic surface $M \subset \mathbb{E}^3$ depend on two arbitrary functions of one variable ([4], Part 2, Problem 5). ∎

5.5 Spaces whose scalar curvature is constant along principal geodesics

In [10] and [3], Chapter 6, one constructs a class of (singular) *parabolic semi-symmetric* spaces for which the scalar curvature is constant along each principal geodesic. This cannot happen for the pseudo-symmetric spaces of nonzero constant type as the following Theorem shows.

Theorem 5.13. *Let* (M, g) *be a three-dimensional foliated pseudo-symmetric space, and let the scalar curvature* $\mathrm{Sc}(g)$ *be constant along each principal geodesic. Then* (M, g) *is a non-elliptic space of type* (H) *or an elliptic space of type* (E).

Proof. Because the scalar curvature $\mathrm{Sc}(g)$ is given, in the hyperbolic case, by

$$\mathrm{Sc}(g) = 2k + 4\tilde{c} = \frac{2\sigma}{fA} - 6\lambda^2 = \frac{2\sigma}{f_1 \cosh(\lambda y) + f_2 \sinh(\lambda y) + f_3} - 6\lambda^2$$

and, in the elliptic case, by

$$\begin{aligned}\mathrm{Sc}(g) = 2k + 4\tilde{c} &= \frac{2\sigma}{fA} + 6\lambda^2 \\ &= \frac{2\sigma}{f_1 \cos(2\lambda y) + f_2 \sin(2\lambda y) + f_3} + 6\lambda^2,\end{aligned}$$

our condition means that $f_1 = f_2 = 0$, and hence $\Delta' = -\epsilon 4\lambda^2 {f_3}^2$ due to (4.5). Hence, according to Proposition 4.7, the assertion follows. ∎

Remark. For some detailed study on our spaces whose scalar curvature $\mathrm{Sc}(g)$ is constant along each principal geodesic, we refer to [14] and [13].

6 The quasi-explicit classification of spaces of elliptic type

The spaces of elliptic type are much more difficult to deal with because the coefficients A, f and C in (1.8) cannot be expressed in general in the form of linear combinations of $\cosh(\lambda y)$ and $\sinh(\lambda y)$; or of $\cos(\lambda y)$ and $\sin(\lambda y)$. We are not able to solve the classification problem explicitly, but we can still prove the local isometry classes of metrics depend on essentially three arbitrary functions of two variables. Also, we give an example of explicit family of metrics depending on two arbitrary functions of two variables.

We see first that the functions a_0 and φ_0 are always nonzero on a space of type (E) (*cf.* (4.6)). Also, we must have $h \neq 0$. (If $h = 0$, then $b = c$ in (3.3) and hence $\Delta \geq 0$ in Proposition 4.3.) From (2.24he) and (2.5he) we see that

(6.1) $\epsilon({a_1}^2 - {a_3}^2) + {a_2}^2 < 0, \quad \epsilon({\varphi_1}^2 - {\varphi_3}^2) + {\varphi_2}^2 < 0,$

and, from (4.5), we have

(6.2) $\epsilon({f_1}^2 - {f_3}^2) + {f_2}^2 < 0.$

We start with the following simplification:

Proposition 6.1. *Every metric g of type* (E) *can be expressed locally, using the convenient coordinates and convenient coframe, in the form* (1.8)*, where $f_2 = 0$, $a_2 \neq 0$ and $b_2 = 0$.*

The *proof* is a modification of that of Proposition 8.1 from [10] using the fact that fA/f_3 is a Riemannian invariant (*cf.* (3.14he)). Notice that the two cases in Proposition 8.1 from [10] are reduced to one case only. In fact, if $f_2 = a_2 = 0$, then, making substitution $\bar{y} = y + 1$, we have $\bar{a}_2 \neq 0$.

Now we study the “fine structure” of the partial differential equations

(A1) $(A\alpha)'_y + \beta'_x = 0$

and

(A2) $R'_y - \beta'_w = 0$

with $\beta \neq 0$, that is, we shall write (A1) and (A2) as a system of partial differential equations for functions of two variables only.

We substitute into (A1) the function $A\alpha$ in the form

$$A\alpha = \frac{1}{2fA}\left[(A^2)'_w - 2(AC)'_x + \frac{AC}{A^2}(A^2)'_x - H(A^2)'_y\right],$$

which follows from (1.12) and (1.10), and the function β in the form $\beta = \lambda a_0/A^2$ (see (2.22)). Taking the common denominator $2A^4(fA)^2$ and using (2.6he), (2.23he) and (2.28he), respectively, we obtain the numerator of the left-hand side of the equation (A1) as a linear combination of c^3, c^2s, c^2, cs, c, s and 1, where $c = \cosh(2\lambda y)$ and $s = \sinh(2\lambda y)$ in the hyperbolic case; $c = \cos(2\lambda y)$ and $s = \sin(2\lambda y)$ in the elliptic case. Each coefficient of this linear combination depends on w and x only, and thus it must vanish if (A1) is satisfied. This gives seven partial differential equations which are linear with respect to a'_{0x}, a'_{1x}, a'_{2x}, a'_{3x}, V_1, V_2 and V_3, where

$$(6.3)\quad \begin{cases} V_1 = a'_{1w} - 2b'_{1x} - 2\lambda H a_2, \\ V_2 = a'_{2w} - 2b'_{2x} + \epsilon 2\lambda H a_1, \\ V_3 = a'_{3w} - 2b'_{3x}. \end{cases}$$

Using the formula (2.24he) in the form

(6.4) $a_0{}^2 = -\epsilon(a_1{}^2 - a_3{}^2) - a_2{}^2$

and its derivative

(6.5) $a_0 a'_{0x} = -\epsilon(a_1 a'_{1x} - a_3 a'_{3x}) - a_2 a'_{2x}$,

we can eliminate the derivative a'_{0x} in all equations. We obtain the final form of the equation (A1) as the system of partial differential equations

$$(6.6)\quad \sum_{i=1}^{3} a_0 P^i_\alpha V_i + \sum_{i=1}^{3} Q^i_\alpha a'_{ix} = 0, \quad \alpha = 1, 2, \ldots, 7,$$

where

$$\begin{aligned}
&P^1_1 = 2a_1a_2f_3, && P^2_1 = (a_1{}^2 - \epsilon a_2{}^2)f_3, && P^3_1 = -2a_1a_2f_1, \\
&P^1_2 = (a_1{}^2 - \epsilon a_2{}^2)f_3, && P^2_2 = -2\epsilon a_1a_2f_3, && P^3_2 = -(a_1{}^2 - \epsilon a_2{}^2)f_1, \\
&P^1_3 = 2a_2a_3f_3, && P^2_3 = (a_1{}^2 - \epsilon a_2{}^2)f_1 + 2a_1a_3f_3, && P^3_3 = -2a_2a_3f_1, \\
&P^1_4 = a_1a_3f_3, && P^2_4 = -\epsilon a_2(a_1f_1 + a_3f_3), && P^3_4 = -a_1a_3f_1, \\
&P^1_5 = 2a_1a_2f_3, && P^2_5 = -2a_1a_3f_1 - (\epsilon a_2{}^2 + a_3{}^2)f_3, && P^3_5 = -2a_1a_2f_1, \\
&P^1_6 = (a_2{}^2 + \epsilon a_3{}^2)f_3, && P^2_6 = -2a_2a_3f_1, && P^3_6 = -(a_2{}^2 + \epsilon a_3{}^2)f_1, \\
&P^1_7 = 2a_2a_3f_3, && P^2_3 = -(\epsilon a_2{}^2 + a_3{}^2)f_1, && P^3_7 = -2a_2a_3f_1,
\end{aligned}$$

$$Q^1_1 = -a_0a_2b_3f_1 + a_0a_2b_1f_3 + ({a_2}^2 - \epsilon {a_3}^2){f_1}^2,$$

$$Q^1_2 = -a_0(a_1b_3 - a_3b_1)f_1 + a_0a_1b_1f_3 + a_1a_2{f_1}^2,$$

$$Q^1_3 = -a_0a_2b_1f_1 - \epsilon a_1a_3{f_1}^2 + 2({a_2}^2 - \epsilon {a_3}^2)f_1f_3,$$

$$Q^1_4 = -a_0a_3b_1f_3 + a_1a_2f_1f_3,$$

$$Q^1_5 = 2a_0a_2b_1f_3 - ({a_2}^2 - \epsilon {a_3}^2){f_3}^2 + 2\epsilon a_1a_3f_1f_3,$$

$$Q^1_6 = \epsilon a_0a_3b_3f_3 + \epsilon a_1a_2{f_3}^2,$$

$$Q^1_7 = a_0a_2b_3f_3 + \epsilon a_1a_3{f_3}^2,$$

$$Q^2_1 = -a_0(a_1b_3 - a_3b_1)f_1 + a_0a_1b_1f_3 - a_1a_2{f_1}^2,$$

$$Q^2_2 = \epsilon a_0a_2b_3f_1 - \epsilon a_0a_2b_1f_3 - ({a_1}^2 - {a_3}^2){f_1}^2,$$

$$Q^2_3 = a_0a_1b_1f_1 + 2a_0a_3b_1f_3 - a_2a_3{f_1}^2 - 2a_1a_2f_1f_3,$$

$$Q^2_4 = -({a_1}^2 - {a_3}^2)f_1f_3,$$

$$Q^2_5 = -2a_0a_1b_3f_1 - a_0a_3b_3f_3 + a_1a_2{f_3}^2 + 2\epsilon a_2a_3f_1f_3,$$

$$Q^2_6 = -a_0a_2b_3f_1 + a_0a_2b_1f_3 - ({a_1}^2 - {a_3}^2){f_3}^2,$$

$$Q^2_7 = -a_0a_3b_3f_1 - a_0(a_1b_3 - a_3b_1)f_3 + \epsilon a_2a_3{f_3}^2,$$

$$Q^3_1 = -a_0a_2b_1f_1 + \epsilon a_1a_3{f_1}^2,$$

$$Q^3_2 = -a_0a_1b_1f_1 - a_2a_3{f_1}^2,$$

$$Q^3_3 = -2a_0a_2b_3f_1 + (\epsilon {a_1}^2 - {a_3}^2){f_1}^2 + 2\epsilon a_1a_3f_1f_3,$$

$$Q^3_4 = -a_0a_1b_3f_1 - a_2a_3f_1f_3,$$

$$Q^3_5 = a_0a_2b_3f_3 - \epsilon a_1a_3{f_3}^2 + 2(\epsilon {a_1}^2 + {a_2}^2)f_1f_3,$$

$$Q^3_6 = -a_0a_3b_3f_1 - \epsilon a_0(a_1b_3 - a_3b_1)f_3 - \epsilon a_2a_3f_1{f_3}^2,$$

$$Q^3_7 = -a_0a_2b_3f_1 + a_0a_2b_1f_3 - (\epsilon {a_1}^2 + {a_2}^2){f_3}^2.$$

Next, we substitute into (A2) the function R in the form

$$R = \frac{1}{2fA}\left[(f^2 + C^2)'_x + H(h + (AC)'_y) - \frac{AC}{A^2}(A^2)'_w\right],$$

which we obtain from $(1.13)_1$, (1.12) and (1.10), and the function β in the form $\beta = \lambda a_0/A^2$. By the same argument as that for the previous equation (A1), we obtain once more seven partial differential equations. They are now linear with respect to a'_{0w}, a'_{1w}, a'_{2w}, a'_{3w}, W_1, W_2 and W_3, where

$$(6.7)\quad \begin{cases} W_1 = -\epsilon\,\dfrac{1}{\lambda}\varphi'_{1x} + 2\lambda H b_2, \\ W_2 = \dfrac{1}{\lambda}\varphi'_{2x} + 2\epsilon\lambda H b_1, \\ W_3 = \dfrac{1}{\lambda}\varphi'_{3x} + Hh. \end{cases}$$

Using (6.4) and the formula for a'_{0w} similar to (6.5), we can also eliminate the derivative a'_{0w} in all equations. We obtain the final form of the equation (A2) as a system of partial differential equations analogous to (6.6):

$$(6.8)\quad \sum_{i=1}^{3} a_0 P^i_\alpha W_i - \sum_{i=1}^{3} Q^i_\alpha a'_{iw} = 0, \quad \alpha = 1, 2, \ldots, 7.$$

The following proposition will be crucial for reducing our partial differential equations to essentially independent ones.

Proposition 6.2. *The rank of the matrix* $\left[P^i_\alpha,\ Q^i_\alpha\right]$ *is at most two.*

Proof. Since $a_2 \neq 0$ and $b_2 = f_2 = 0$, we have from (2.51)

$$(6.9)\quad \varphi_4 = \epsilon\,\frac{\lambda(a_1 f_3 - a_3 f_1)}{a_2},$$

and hence we have

$$(6.10)\quad \begin{cases} b_1 = \dfrac{{a_2}^2 f_3 + \epsilon a_1(a_1 f_3 - a_3 f_1)}{a_0 a_2}, \\ b_3 = \dfrac{{a_2}^2 f_1 + \epsilon a_3(a_1 f_3 - a_3 f_1)}{a_0 a_2}. \end{cases}$$

Substituting from (6.10) for b_1 and b_3 in the entries of the matrix $\left[Q^i_\alpha\right]$, we see that

$$\left[P^3_\alpha\right] = -\frac{f_1}{f_3}\left[P^1_\alpha\right],$$

$$\left[Q^1_\alpha\right] = \epsilon\,\frac{a_1 f_3 - a_3 f_1}{a_2}\left[P^1_\alpha\right] - \epsilon f_3\left[P^2_\alpha\right],$$

$$\left[Q^2_\alpha\right] = -\frac{{f_1}^2 - {f_3}^2}{f_3}\left[P^1_\alpha\right] + \epsilon\,\frac{a_1 f_3 - a_3 f_1}{a_2}\left[P^2_\alpha\right],$$

$$\left[Q^3_\alpha\right] = -\epsilon\,\frac{f_1(a_1 f_3 - a_3 f_1)}{a_2 f_3}\left[P^1_\alpha\right] + \epsilon f_1\left[P^2_\alpha\right],$$

which prove the assertion. ∎

Corollary 6.3. *Each system of partial differential equations* (6.6) *or* (6.8) *contains at most two linearly independent equations.*

Thus, the equations (A1) and (A2) are essentially reduced to four partial differential equations in two variables. We shall see later that, as in [10], we can make an additional reduction to only two equations (one of the form (6.6) and one of the form (6.8)).

Proposition 6.4. *The following algebraic formulas are consequences of the algebraic equations from Theorem* 2.8 *and of the assumptions of Proposition* 6.1:

(6.11) $\varphi_1 = \nu a_1, \quad \varphi_2 = \epsilon\nu a_2, \quad \varphi_3 = -\epsilon\nu a_3,$

where

(6.12) $\nu = \dfrac{\lambda[{a_2}^2({f_1}^2 - {f_3}^2) - \epsilon(a_1f_3 - a_3f_1)^2]}{{a_0}^2{a_2}^2}, \quad \nu = \epsilon\,\dfrac{\varphi_0}{a_0},$

(6.13) ${a_0}^2 = -\epsilon({a_1}^2 - {a_3}^2) - {a_2}^2.$

Further, $f_2 = 0$ *and*

(6.14) $\begin{cases} b_1 = \dfrac{{a_2}^2f_3 + \epsilon a_1(a_1f_3 - a_3f_1)}{a_0a_2}, \quad b_2 = 0, \\ b_3 = \dfrac{{a_2}^2f_1 + \epsilon a_3(a_1f_3 - a_3f_1)}{a_0a_2}, \end{cases}$

(6.15) $h = -\epsilon\,\dfrac{2\lambda(a_1f_1 - a_3f_3)}{a_0}, \quad \varphi_4 = \epsilon\,\dfrac{\lambda(a_1f_3 - a_3f_1)}{a_2}, \quad \varphi_5 = 2\varphi_4.$

Conversely, if a_1*,* a_2*,* a_3*,* f_1 *and* f_3 *are arbitrary functions, and if the other basic functions are defined as above, then all algebraic equations of Theorem* 2.8 *hold.*

Proof. We show only the necessity of (6.11)–(6.15). The sufficiency will be proved by the direct check. The equations $(2.44)_3$ and $(2.44)_5$ imply $a_1\varphi_2 - \epsilon a_2\varphi_1 = 0$ and $a_2\varphi_3 + a_3\varphi_2 = 0$. Hence the formulas (6.11) hold with some function $\nu = \nu(w, x)$ of the variables w and x. Substituting (6.11) and $(6.10)_1$ into $(2.44)_1$, and using (2.24he), we obtain $(6.12)_1$. The formula (6.13) is a direct consequence of (2.24he). The formulas $(6.14)_{1,3}$ and $(6.15)_2$ follow from $b_2 = f_2 = 0$ as shown in the proof of Proposition 6.2. Next, from (6.11), (2.5he) and (2.24he), we have ${\varphi_0}^2 = \nu^2{a_0}^2$. Here, the relation (4.5) implies that $\epsilon({f_1}^2 - {f_3}^2)$ is negative because the discriminant Δ' is negative, hence $\epsilon\nu$ is negative. On the other hand, (4.4) together with (4.5) implies that $a_0\varphi_0$ is negative. Hence we obtain $(6.12)_2$. We obtain $(6.15)_1$ from (2.47) and $f_2 = 0$. Finally, $(6.15)_3$ is the same as (2.48). ∎

We need later the relation

(6.16) $\nu = \dfrac{\lambda(f_1b_3 - f_3b_1)}{a_0a_1},$

which follows from (6.14) and (6.12).

Now let us return to the system of partial differential equations (6.6) and (6.8). Specifying Corollary 6.3, we see easily that the system (6.6) reduces to two partial differential equations

(6.17) $a_0V_2 - \epsilon f_3a'_{1x} + \epsilon\,\dfrac{a_1f_3 - a_3f_1}{a_2}a'_{2x} - f_1a'_{3x} = 0$

and

$$a_0 f_3 V_1 + \frac{a_0(a_1 f_3 - a_3 f_1)}{a_2} V_2 - a_0 f_1 V_3 \tag{6.18}$$
$$+ \left[\epsilon \left(\frac{a_1 f_3 - a_3 f_1}{a_2} \right)^2 - \left({f_1}^2 - {f_3}^2 \right) \right] a'_{2x} = 0.$$

The system (6.8) reduces to two analogous equations

$$a_0 W_2 + \epsilon f_3 a'_{1w} - \epsilon \frac{a_1 f_3 - a_3 f_1}{a_2} a'_{2w} + f_1 a'_{3w} = 0 \tag{6.19}$$

and

$$a_0 f_3 W_1 + \frac{a_0(a_1 f_3 - a_3 f_1)}{a_2} W_2 - a_0 f_1 W_3 \tag{6.20}$$
$$- \left[\epsilon \left(\frac{a_1 f_3 - a_3 f_1}{a_2} \right)^2 - \left({f_1}^2 - {f_3}^2 \right) \right] a'_{2w} = 0.$$

Using (6.3), (6.7), (6.11), (6.14) and (6.16), we see, after lengthy but routine calculations, that (6.18) and (6.20) *are consequences* of (6.17) and (6.19).

Substituting $(6.3)_2$ and $(6.7)_2$ into (6.17) and (6.19), respectively, and using $(6.11)_2$, we have

$$\begin{cases} a_0 a'_{2w} + \epsilon 2\lambda H a_0 a_1 - \epsilon a_2 f_3 \left(\dfrac{a_1}{a_2} \right)'_x + \epsilon a_2 f_1 \left(\dfrac{a_3}{a_2} \right)'_x = 0, \\ a_0 (v a_2)'_x - 2\lambda^2 H a_0 b_1 + \lambda a_2 f_3 \left(\dfrac{a_1}{a_2} \right)'_w - \lambda a_2 f_1 \left(\dfrac{a_3}{a_2} \right)'_w = 0. \end{cases} \tag{6.21}$$

Further, due to $(6.15)_1$, we have the relation

$$2\lambda(a_1 f_1 - a_3 f_3) = -\epsilon a_0 H'_x. \tag{6.22}$$

Introducing new functions $u = u(w, x)$ and $v = v(w, x)$ of the variables w and x such that

$$a_1 = u a_2, \quad a_3 = v a_2, \quad -\epsilon(u^2 - v^2) > 0, \tag{6.23}$$

we rewrite (6.21) in the form

$$\begin{cases} a_0 a'_{2w} + \epsilon 2\lambda H a_0 a_1 - \epsilon a_2 f_3 u'_x + \epsilon a_2 f_1 v'_x = 0, \\ a_0 (v a_2)'_x - 2\lambda^2 H a_0 b_1 + \lambda a_2 f_3 u'_w - \lambda a_2 f_1 v'_w = 0. \end{cases} \tag{6.24}$$

Here, from (6.12)–(6.14), we get

$$\begin{cases} a_0 = \sqrt{-\epsilon(u^2 - v^2) - 1}\, a_2, \\ b_1 = \dfrac{f_3 + \epsilon u(u f_3 - v f_1)}{\sqrt{u^2 - v^2 - 1}}, \\ \nu = \dfrac{\lambda \left[{f_1}^2 - {f_3}^2 - \epsilon (u f_3 - v f_1)^2 \right]}{(u^2 - v^2 - 1)\, {a_2}^2}, \end{cases} \tag{6.25}$$

where we normalize the signs of a_2 and a_0 to make them positive.

Let now u, v and H be arbitrary analytic functions. Substituting for a_0 from $(6.25)_1$ into (6.22) and into $(6.24)_1$, and solving them with respect to f_1 and f_3, we can express f_1 and f_3 in the form

$$(6.26)\quad \begin{cases} f_1 = g_1 a'_{2w} + g_2 a_2 + g_3, \\ f_3 = h_1 a'_{2w} + h_2 a_2 + h_3, \end{cases}$$

where g_i's and h_i's are known functions. Substituting (6.26) into $(6.24)_2$ which has been transformed by (6.25), we obtain a partial differential equation of the form

$$(6.27)\quad a''_{2wx} = \Psi(a'_{2w}, a'_{2x}, a_2, w, x),$$

where Ψ is a fixed analytic function of five variables. The general solution of (6.27) depends on two arbitrary (analytic) functions of one variable. Thus, *the generic family of metrics of type* (E) *depends on three arbitrary functions of two variables,* namely, u, v and H.

Now, we can go further and prove that even *the local isometry classes* of our metrics still depend essentially on three functions. The proof is a modification of that of Theorem 8.5 from [10]. We use the fact that fA/f_3 is a Riemannian invariant (see (3.14he)) and that the hyperbolic cosine function and the cosine function are even functions.

Theorem 6.5. *The local isometry classes of metrics of type* (E) *are parameterized by three arbitrary functions of two variables modulo two arbitrary functions of one variable.*

The equation (6.27) can not be solved explicitly, in general. Yet, we give here an explicit family of the metrics of type (E).

Example 6.6. Consider the "singular" case $a_2 = 0$ of Proposition 6.4. Then we have

$$(6.28)\quad \begin{cases} \varphi_1 = \nu a_1, \quad \varphi_2 = a_2 = 0, \quad \varphi_3 = -\epsilon \nu a_3, \\ \varphi_0 = \epsilon \nu a_0, \quad \varphi_5 = 2\varphi_4 \end{cases}$$

and

$$(6.29)\quad b_2 = f_2 = 0.$$

From $(2.45)_2$ we see that there is a function $\xi = \xi(w, x)$ of the variables w and x such that

$$(6.30)\quad f_1 = \xi a_1, \quad f_3 = \xi a_3.$$

Hence, using (6.22) and (2.51), we have

$$(6.31)\quad a_0 h = -\epsilon 2\lambda \xi ({a_1}^2 - {a_3}^2),$$

$$(6.32)\quad b_1 = \frac{a_1 \varphi_4}{\lambda a_0}, \quad b_3 = \frac{a_3 \varphi_4}{\lambda a_0}.$$

Finally, we have

$$(6.33)\quad {a_0}^2 = -\epsilon({a_1}^2 - {a_3}^2),$$

and, from $(2.43)_1$ or $(2.44)_1$, we deduce

$$\frac{{\varphi_4}^2}{\lambda {a_0}^2} + \lambda\xi^2 = -\epsilon\nu. \tag{6.34}$$

Here a_1, a_3, ξ and φ_4 are arbitrary functions of the variables w and x. Conversely, if a_1, a_3, ξ and φ_4 are arbitrary functions of the variables w and x, and if the other basic functions are given by (6.28)–(6.34), then all algebraic equations mentioned in Theorem 2.8 are satisfied.

In addition, from (6.31) and (6.33), we get

$$h = 2\lambda\xi a_0, \quad h = H'_x. \tag{6.35}$$

Further, a careful check shows that the system of partial differential equations (6.6) and (6.8) can be now reduced, instead of the form (6.21), to the form

$$a_0 V_2 - \epsilon(f_3 a'_{1x} - f_1 a'_{3x}) = 0, \tag{6.36}$$

$$a_0 W_2 + \epsilon(f_3 a'_{1w} - f_1 a'_{3w}) = 0. \tag{6.37}$$

All other partial differential equations are consequences of (6.36) and (6.37). Putting $U = a_3/a_1$, we can rewrite (6.36) and (6.37) in the form

$$2\lambda H a_0 + \xi a_1 U'_x = 0, \tag{6.38}$$

$$2H\varphi_4 + \xi a_1 U'_w = 0. \tag{6.39}$$

Then we have the following explicit family of solutions satisfying the equations (6.38) and (6.39) and the condition (6.35). Choose U and H as arbitrary functions of the variables w and x, and put

$$\begin{cases} a_1 = -\epsilon \dfrac{hU'_x}{4\lambda^2 H(U^2-1)}, \quad a_3 = a_1 U, \quad a_0 = a_1\sqrt{\epsilon(U^2-1)}, \\ \xi = -\dfrac{2\lambda H\sqrt{\epsilon(U^2-1)}}{U'_x}, \quad \varphi_4 = -\dfrac{hU'_w}{4\lambda H\sqrt{\epsilon(U^2-1)}}, \quad h = H'_x. \end{cases} \tag{6.40}$$

Here we always assume $U'_x \neq 0$ and $\epsilon(U^2-1) > 0$. (Also, we normalize the signs of a_1, a_3 and a_0 to make them all positive.) Then the function ν is calculated from (6.34) and remaining coefficients are given by (6.28)–(6.30) and (6.32). This defines the wanted class of metrics.

References

[1] E. Boeckx, Foliated semi-symmetric spaces, Doctoral Thesis, Leuven, 1994.

[2] E. Boeckx, O. Kowalski and L. Vanhecke, Non-homogeneous relatives of symmetric spaces, *Diff. Geom. Appl.* **4**(1994), 45–69.

[3] E. Boeckx, O. Kowalski and L. Vanhecke, *Riemannian Manifolds of Conullity Two*, World Scientific, Singapore, 1996.

[4] E. Cartan, *Les systèmes différentiels extérieurs et leurs applications geometriques,* 2nd edition, Hermann, Paris, 1971.

[5] J. Deprez, R. Deszcz and L. Verstraelen, Examples of pseudo-symmetric conformally flat warped products, *Chinese J. Math.* **17**(1989), 51–65.

[6] R. Deszcz, On pseudo-symmetric spaces, *Bull. Soc. Math. Belgium,* Série A. **44**(1992), 1–34.

[7] V. Hájková, Foliated semi-symmetric spaces in dimension 3 (in Czech), Doctoral Thesis, Prague, 1995.

[8] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry* I, Interscience Publishers, New York-London-Sydney, 1963.

[9] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry* II, Interscience Publishers, New York-London-Sydney, 1969.

[10] O. Kowalski, An explicit classification of 3-dimensional Riemannian spaces satisfying $R(X, Y) \cdot R = 0$, *Czechoslovak Math. J.* **46(121)**(1996), 427–474. (Preprint 1991).

[11] O. Kowalski, A classification of Riemannian 3-manifolds with constant principal Ricci curvatures $\rho_1 = \rho_2 \neq \rho_3$, *Nagoya Math. J.* **132**(1993), 1–36.

[12] O. Kowalski and S. Ž. Nikčević, On Ricci eigenvalues of locally homogeneous Riemannian 3-manifolds, *Geometriae Dedicata* **62**(1996), 65–72.

[13] O. Kowalski and M. Sekizawa, Local isometry classes of Riemannian 3-manifolds with constant Ricci eigenvalues $\rho_1 = \rho_3 \neq \rho_3$, *Archivum Math.* (*Brno*) **32**(1996), 137–145.

[14] O. Kowalski and M. Sekizawa, Riemannian 3-manifolds with c-conullity two, *Bolletino U. M. I.* (7)**11**-B(1997), Suppl. fasc. 2, 161–184.

[15] O. Kowalski and M. Sekizawa, Three-dimensional Riemannian manifolds of c-conullity two, Chapter 11 of "Riemannian Manifolds of Conullity Two", World Scientific, Singapore, 1996.

[16] O. Kowalski and M. Sekizawa, Pseudo-symmetric spaces of constant type in dimension three—elliptic spaces, *Rendiconti di Matematica, Serie VII,* Volume **17**, Roma(1997), 477–512.

[17] O. Kowalski and M. Sekizawa, Pseudo-symmetric spaces of constant type in dimension three—non-elliptic spaces, *Bull. Tokyo Gakugei Univ.* (4) **50**(1998), 1–28.

[18] O. Kowalski, F. Tricerri and L. Vanhecke, Exemples nouveaux de variétés riemanniennes non homogènes dont le tenseur de courbure est celui d'un espace symétrique riemannien, *C. R. Acad. Sci. Paris Ser.* I **311**(1990), 355–360.

[19] O. Kowalski, F. Tricerri and L. Vanhecke, Curvature homogeneous Riemannian manifolds, *J. Math. Pures Appl.* **71**(1992), 471–501.

[20] Z. I. Szabó, Structure theorems on Riemannian manifolds satisfying $R(X, Y) \cdot R = 0$, I, Local version, *J. Diff. Geom.* **17**(1982), 531–582.

[21] Z. I. Szabó, Structure theorems on Riemannian manifolds satisfying $R(X, Y) \cdot R = 0$ II, Global version, *Geom. Dedicata* **19**(1985), 65–108.

[22] Z. I. Szabó, Classification and construction of complete hypersurfaces satisfying $R(X, Y) \cdot R = 0$, *Acta. Sci. Math.*(Hung.) **47**(1984), 321–348.

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