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Clasp-pass moves on knots, links and spatial graphs

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Dedicated to Professor Kazuaki Kobayashi in honor of his 60th birthday

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Abstract

A *clasp-pass move* is a local move on oriented links introduced by Habiro in 1993. He showed that two knots are transformed into each other by clasp-pass moves if and only if they have the same second coefficient of the Conway polynomial. We extend his classification to two-component links, three-component links, algebraically split links, and spatial embeddings of a planar graph that does not contain disjoint cycles. These are classified in terms of linking numbers, the second coefficient of the Conway polynomial, the Arf invariant, and the Milnor μ-invariant. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Throughout this paper we work in the piecewise linear category. The following result is the first nontrivial geometric classification of all oriented knots by an algebraic invariant.

Theorem 1.1 (Kauffman [5]). *Let K_1 and K_2 be oriented knots in the three-sphere S^3. Then K_1 and K_2 are transformed into each other by pass moves and ambient isotopy if and only if $a_2(K_1) \equiv a_2(K_2) \pmod 2$ where a_i denotes the ith coefficient of the Conway polynomial, and a pass move is a local move on oriented knots as illustrated in Fig.* 1.

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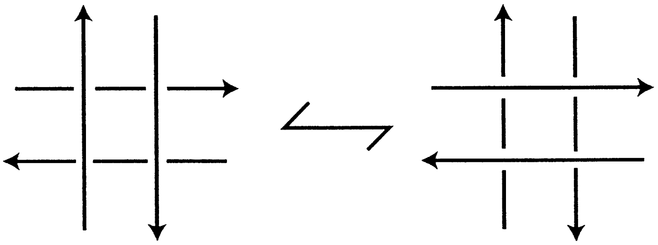


Fig. 1.

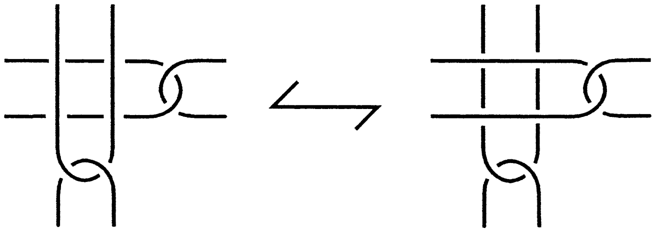


Fig. 2.

In 1993 Habiro defined a *clasp-pass move* on knots and links as a local move illustrated in Fig. 2 and showed the following theorem. In [3] Habiro shows Theorem 1.2 as a corollary to his original *clasper theory*. A direct proof is given in [2].

Theorem 1.2 [2], [3, Proposition 7.1]. *Let K_1 and K_2 be oriented knots in S^3. Then K_1 and K_2 are transformed into each other by clasp-pass moves and ambient isotopy if and only if $a_2(K_1) = a_2(K_2)$.*

Note that a clasp-pass move is always realized by a pass move no matter how the strings are oriented. Therefore Theorem 1.2 is a refinement of Theorem 1.1. Namely Theorem 1.1 gives a Z_2 classification of oriented knots whereas Theorem 1.2 gives a Z classification of oriented knots.

In this paper we consider the classification of certain links and spatial graphs under clasp-pass moves. A *delta move* is a local move defined in [11] as illustrated in Fig. 3. It is not hard to see that the mirror image move of a delta move is realized by a delta move and ambient isotopy. However we do not use this fact later.

A *delta equivalence* is an equivalence relation generated by delta moves and ambient isotopy. Similarly a *clasp-pass equivalence* is an equivalence relation generated by clasp-pass moves and ambient isotopy. We will show in Section 2 that a clasp-pass move is realized by two applications of a delta move. So clasp-pass equivalence is stronger than delta equivalence. Therefore before considering the clasp-pass classification we must consider the delta classification. The following delta classification of links is known.

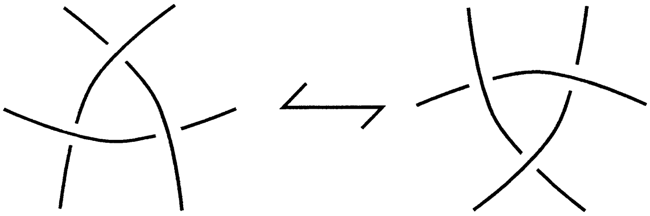


Fig. 3.

Theorem 1.3 [11]. *Let $L = J_1 \cup \cdots \cup J_n$ and $M = K_1 \cup \cdots \cup K_n$ be ordered oriented n-component links. Then L and M are delta equivalent if and only if $\ell k(J_i, J_j) = \ell k(K_i, K_j)$ $(1 \leqslant i < j \leqslant n)$, where ℓk denotes the linking number.*

We will give a new proof of Theorem 1.3 in Section 2. The authors believe our proof is simpler than the prior proof by Murakami and Nakanishi [11]. Under these situations we extend Theorem 1.2 as follows.

A link L is called *algebraically split* if every 2-component sublink of L has linking number zero. Note that by Theorem 1.3 the algebraically split links are exactly the links that are delta equivalent to a trivial link.

Theorem 1.4. *Let $L = J_1 \cup \cdots \cup J_n$ and $M = K_1 \cup \cdots \cup K_n$ be ordered oriented n-component algebraically split links. Then the following conditions are equivalent*:

(1) *L and M are clasp-pass equivalent,*

(2) *$a_2(J_i) = a_2(K_i)$ $(1 \leqslant i \leqslant n)$, $a_3(J_i \cup J_j) \equiv a_3(K_i \cup K_j) \pmod 2$ $(1 \leqslant i < j \leqslant n)$ and $\mu(J_i \cup J_j \cup J_k) = \mu(K_i \cup K_j \cup K_k)$ $(1 \leqslant i < j < k \leqslant n)$,*

(3) *$a_2(J_i) = a_2(K_i)$ $(1 \leqslant i \leqslant n)$, $\mathrm{Arf}(J_i \cup J_j) = \mathrm{Arf}(K_i \cup K_j)$ $(1 \leqslant i < j \leqslant n)$ and $\mu(J_i \cup J_j \cup J_k) = \mu(K_i \cup K_j \cup K_k)$ $(1 \leqslant i < j < k \leqslant n)$.*

Here $\mu = \mu_{ijk}$ denotes the Milnor invariant [9] *and* Arf *the Arf invariant* [15].

A link L is *proper* if $\ell k(L - K, K)$ is even for any component K of L. Here $\ell k(L - K, K)$ denotes the sum of the linking numbers of K and other components of L.

Theorem 1.5. *Let $L = J_1 \cup J_2$ and $M = K_1 \cup K_2$ be ordered oriented 2-component links. Then the following conditions are equivalent*:

(1) *L and M are clasp-pass equivalent,*

(2) *$\ell k(J_1, J_2) = \ell k(K_1, K_2)$, $a_2(J_i) = a_2(K_i)$ $(i = 1, 2)$ and $a_3(L) \equiv a_3(M) \pmod 2$,*

(3) *$\ell k(J_1, J_2) = \ell k(K_1, K_2)$, $a_2(J_i) = a_2(K_i)$ $(i = 1, 2)$ and $\mathrm{Arf}(L) = \mathrm{Arf}(M)$ if L and M are proper links.*

Remark 1.6. (1) In addition the following is shown in the proof of Theorem 1.5. If $\ell k(J_1, J_2) = \ell k(K_1, K_2)$ is odd and $a_2(J_i) = a_2(K_i)$ $(i = 1, 2)$ then $a_3(L) \equiv a_3(M) \pmod 2$ and L and M are clasp-pass equivalent. However for any even integer l and integers α_1, α_2 there are L and M such that $\ell k(J_1, J_2) = \ell k(K_1, K_2) = l$, $a_2(J_i) = a_2(K_i) = \alpha_i$ $(i = 1, 2)$ and $a_3(L) \equiv a_3(M) + 1 \pmod 2$.

(2) Let L_0 be an link and $[L_0]_2$ (respectively $[L_0]_3$) the delta (respectively clasp-pass) equivalence class that contains L_0. In [20], the authors showed that $[L_0]_2/$(clasp-pass equivalence) forms an Abelian group under certain geometric operation with the unit element $[L_0]_3$ and denote it by $\mathcal{G}_3(L_0)$. (In [20], we study more general moves of spatial graphs.) By [20], $\mathcal{G}_3(L_0)$ and $\mathcal{G}_3(L_0')$ are isomorphic if L_0 and L_0' are either delta equivalent or 1-component. By the definition of this group and the proof of Theorem 1.5, we have that $\mathcal{G}_3(K_1 \cup K_2)$ is isomorphic to $Z \oplus Z \oplus Z_2$ (respectively $Z \oplus Z$) if $\ell k(K_1, K_2)$ is even (respectively odd). Thus there are 2-component links L_0 and L_0' such that $\mathcal{G}_3(L_0)$ and $\mathcal{G}_3(L_0')$ are not isomorphic.

Theorem 1.7. *Let $L = J_1 \cup J_2 \cup J_3$ and $M = K_1 \cup K_2 \cup K_3$ be ordered oriented 3-component links. Then the following conditions are equivalent*:

(1) *L and M are clasp-pass equivalent,*

(2) $\ell k(J_i, J_j) = \ell k(K_i, K_j)$ $(1 \leqslant i < j \leqslant 3)$, $a_2(J_i) = a_2(K_i)$ $(i = 1, 2, 3)$, $a_3(J_i \cup J_j) \equiv a_3(K_i \cup K_j) \pmod 2$ $(1 \leqslant i < j \leqslant 3)$, $\mu(L) \equiv \mu(M)$ $(\mathrm{mod}\ \ell k(J_1, J_2), \ell k(J_2, J_3), \ell k(J_3, J_1))$ *and* $a_4(L) \equiv a_4(M) \pmod 2$,

(3) $\ell k(J_i, J_j) = \ell k(K_i, K_j)$ $(1 \leqslant i < j \leqslant 3)$, $a_2(J_i) = a_2(K_i)$ $(i = 1, 2, 3)$, $\mathrm{Arf}(J_i \cup J_j) = \mathrm{Arf}(K_i \cup K_j)$ *if $J_i \cup J_j$ and $K_i \cup K_j$ are proper links,* $\mathrm{Arf}(L) = \mathrm{Arf}(M)$ *if L and M are proper links and* $\mu(L) \equiv \mu(M)$ $(\mathrm{mod}\ \ell k(J_1, J_2), \ell k(J_2, J_3), \ell k(J_3, J_1))$.

Here $\mu = \mu_{123}$ denotes the Milnor invariant that is defined modulo the greatest common divisor of $\ell k(J_1, J_2)$, $\ell k(J_2, J_3)$ and $\ell k(J_3, J_1)$.

Remark 1.8. In addition the following is shown in the proof of Theorem 1.7. If $\ell k(J_1, J_2)\ell k(J_2, J_3)\ell k(J_3, J_1)$ is even, $\ell k(J_i, J_j) = \ell k(K_i, K_j)$ $(1 \leqslant i < j \leqslant 3)$, $a_2(J_i) = a_2(K_i)$ $(i = 1, 2, 3)$, $a_3(J_i \cup J_j) \equiv a_3(K_i \cup K_j) \pmod 2$ $(1 \leqslant i < j \leqslant 3)$ and $\mu(L) = \mu(M)$ then $a_4(L) \equiv a_4(M) \pmod 2$ and L and M are clasp-pass equivalent. However for any odd integers l_1, l_2, l_3 and integers $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \lambda$ there are L and M such that $\ell k(J_i, J_{i+1}) = \ell k(K_i, K_{i+1}) = l_i$ $(i = 1, 2, 3)$ (here we consider $3 + 1 = 1$), $a_2(J_i) = a_2(K_i) = \alpha_i$ $(i = 1, 2, 3)$, $a_3(J_i \cup J_{i+1}) \equiv a_3(K_i \cup K_{i+1}) \equiv \beta_i \pmod 2$ $(i = 1, 2, 3)$, $\mu(L) \equiv \mu(M) \equiv \lambda$ $(\mathrm{mod}\ l_1, l_2, l_3)$ and $a_4(L) \equiv a_4(M) + 1 \pmod 2$.

The following is a delta classification of spatial graphs.

Theorem 1.9 [12, Theorems 1.1 and 1.3]. *Two embeddings of a finite graph into S^3 are delta equivalent if and only if they have the same Wu invariant.*

It is known that Wu invariant generalizes linking number [19], and Theorem 1.9 generalizes Theorem 1.3. The following Theorem 1.10 is an immediate consequence of [18, Theorem C] and [12, Theorem 1.3]. A finite graph G is *planar* if it is embeddable into the plane. A *cycle* of a graph G is a subgraph of G that is homeomorphic to a circle.

Theorem 1.10. *For a finite graph G the following conditions are equivalent*:

(1) *Any two embeddings of G into S^3 are delta equivalent,*

(2) *G is a planar graph that does not contain any pair of mutually disjoint cycles.*

Under these situations we consider clasp-pass classification of spatial embeddings of a planar graph that does not contain disjoint cycles.

Theorem 1.11. *Let G be a planar graph that does not contain any pair of mutually disjoint cycles. Let f and g be embeddings of G into S^3. Then f and g are clasp-pass equivalent if and only if $a_2(f(\gamma)) = a_2(g(\gamma))$ for every cycle γ of G.*

Let ω be a *finite type invariant* of order less than or equal to 2 for embeddings of G into S^3 in the sense of [16]. It is known that if two embeddings f and g of G into S^3 are clasp-pass equivalent then $\omega(f) = \omega(g)$ [1,3,13,20]. It is well known that a_2 is a finite type knot invariant of order 2. Let H be a subgraph of a graph G and ω_H a finite type invariant of order less than or equal to k for embeddings of H into S^3. Let ω_G be an invariant for embeddings of G into S^3 defined by $\omega_G(f) = \omega_H(f|_H)$ where $f|_H : H \to S^3$ is a restriction map of $f : G \to S^3$. Then it is easy to see that ω_G is also a finite type invariant of order less than or equal to k. See for example [14]. Hence by Theorem 1.11 we have the following corollary.

Corollary 1.12. *Let G be a planar graph that does not contain any pair of mutually disjoint cycles. Let f and g be embeddings of G into S^3. Then f and g are clasp-pass equivalent if and only if $\omega(f) = \omega(g)$ for any finite type invariant ω of order less than or equal to* 2.

Remark 1.13. The 'if' part of Corollary 1.12 does not hold for two-component links. In fact a_3(Whitehead link) $\equiv 1 \pmod 2$. Hence by Theorem 1.5 the Whitehead link is not clasp-pass equivalent to a trivial link. However by [10] we have that they have the same finite type invariants of order less than or equal to 2. In other words a_3 (mod 2) is not a finite type invariant of order less than or equal to 2.

We remark here that clasp-pass equivalence is related to *surgery equivalence* defined in [7]. In fact it is easy to see that clasp-pass equivalent links are surgically equivalent. However the converse is not true. In fact any two knots are surgically equivalent.

2. Local moves and their equivalence

Let B^3 be the unit 3-ball. We choose and fix an orientation of B^3. A *tangle* T is a disjoint union of finitely many properly embedded arcs in B^3. A tangle T is *trivial* if there is a properly embedded disk in B^3 that contains T.

A *local move* is a pair of trivial tangles (T_1, T_2) with $\partial T_1 = \partial T_2$. Moreover we assume that for each component t of T_1 there is a component u of T_2 such that $\partial t = \partial u$. We say that two local moves (T_1, T_2) and (U_1, U_2) are *equivalent*, denoted by $(T_1, T_2) \cong (U_1, U_2)$, if there is an orientation preserving self-homeomorphism $h : B^3 \to B^3$ such that for each $i = 1, 2$, $h(T_i)$ and U_i are ambient isotopic without moving $h(\partial T_i) = \partial U_i$. A local move

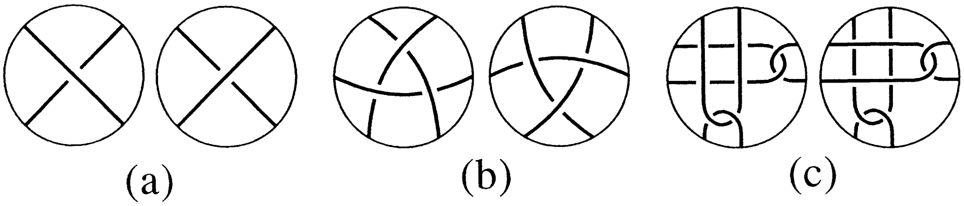


Fig. 4.

that is equivalent to the local move illustrated in Fig. 4(a), (b) or (c) is called a *crossing change*, *delta move* or *clasp-pass move* respectively.

Let (T_1, T_2) be a local move. Then the local move (T_2, T_1) is called the *inverse of* (T_1, T_2). It is easily seen that each of a crossing change, a delta move and a clasp-pass move is equivalent to its inverse.

We choose and fix an orientation of the unit 3-sphere S^3. Let G be a finite graph and $f, g : G \to S^3$ embeddings. Let (T_1, T_2) be a local move that is equivalent to its inverse and $\varphi : B^3 \to S^3$ an orientation preserving embedding. We say that f and g are *related by* (T_1, T_2) *and* φ if the following conditions hold:

(1) if $f(x) \neq g(x)$ then both $f(x)$ and $g(x)$ are contained in $\varphi(\operatorname{int} B^3)$,
(2) $f(G) \cap \varphi(B^3) = \varphi(T_1)$,
(3) $g(G) \cap \varphi(B^3) = \varphi(T_2)$.

Then we also say that g is obtained from f by an *application* of (T_1, T_2). We say that f and g are *related by* (T_1, T_2) if they are related by (T_1, T_2) and φ for some φ. The (T_1, T_2)*-equivalence* is the equivalence relation on the set of all embeddings of G into S^3 generated by the relation above and ambient isotopy. In particular we say that f and g are *delta equivalent* (respectively *clasp-pass equivalent*) if they are transformed into each other by applications of a delta (respectively clasp-pass) move and ambient isotopy. When the graph G is homeomorphic to a disjoint union of n circles, there is a natural correspondence between the ambient isotopy classes of embeddings of G into S^3 and the ambient isotopy classes of ordered oriented n-component links. Under this correspondence we consider (T_1, T_2)-equivalence on ordered oriented links.

For $k = 1, 2$, let $V_k \subset B^3$ be a tangle and $A_k \subset \partial B^3$ a disjoint union of arcs with $\partial V_k = \partial A_k$ as illustrated in Fig. 5. Let $f : G \to S^3$ be an embedding. Let m be a natural number. Let $\psi_i : B^3 \to S^3$ be an orientation preserving embedding for each $i \in \{1, \ldots, m\}$. Let $b_{i,p}$ be a 2-disk embedded in S^3 for each $i \in \{1, \ldots, m\}$ and $p \in \{1, 2\}$ (respectively $p \in \{1, 2, 3\}$). Suppose that $\psi_i(B^3) \cap f(G) = \emptyset$ for each i, $\psi_i(B^3) \cap \psi_j(B^3) = \emptyset$ if $i \neq j$ and $b_{i,p} \cap b_{j,q} = \emptyset$ if $(i, p) \neq (j, q)$. Suppose that $b_{i,p} \cap f(G) = \partial b_{i,p} \cap f(G)$ is an arc away from the vertices of the embedded graph $f(G)$ for each $b_{i,p}$. Suppose that $b_{i,p} \cap \psi_j(B^3) = \emptyset$ if $i \neq j$ and $b_{i,p} \cap \psi_i(B^3) = \partial b_{i,p} \cap \psi_i(\partial B^3)$ is a component of $\psi_i(A_1)$ (respectively $\psi_i(A_2)$) for each $b_{i,p}$. Let $g : G \to S^3$ be an embedding with the following properties:

(1) if $f(x)$ is not contained in $\bigcup_{i,p} b_{i,p}$ then $g(x) = f(x)$,

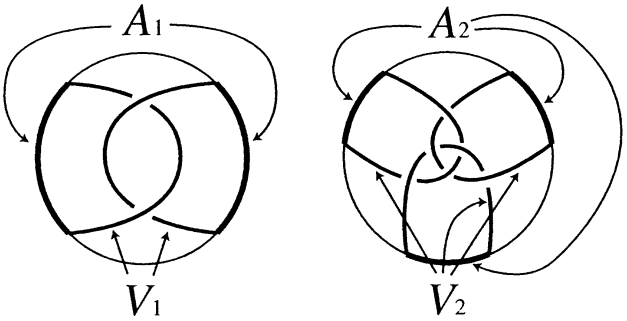


Fig. 5.

(2) $g(G) = f(G) \cup \bigcup_{i,p} \partial b_{i,p} \cup \bigcup_i \psi_i(V_1) - \bigcup_{i,p} \mathrm{int}(f(G) \cap b_{i,p}) - \bigcup_i \psi_i(\mathrm{int}\, A_1)$ (respectively $g(G) = f(G) \cup \bigcup_{i,p} \partial b_{i,p} \cup \bigcup_i \psi_i(V_2) - \bigcup_{i,p} \mathrm{int}(f(G) \cap b_{i,p}) - \bigcup_i \psi_i(\mathrm{int}\, A_2)$).

Then we say that g is a *band sum of Hopf links* (respectively *Borromean rings*) *and* f. We call each $b_{i,j}$ a *band*. Note that each $\psi_i(V_1 \cup A_1)$ (respectively $\psi_i(V_2 \cup A_2)$) is a *Hopf link* (respectively *Borromean ring*) in S^3. The union $b_{i,1} \cup b_{i,2} \cup \psi_i(B^3)$ (respectively $b_{i,1} \cup b_{i,2} \cup b_{i,3} \cup \psi_i(B^3)$) is called a *Hopf chord* (respectively *Borromean chord*) and the bands $b_{i,1}$, $b_{i,2}$ (respectively $b_{i,1}$, $b_{i,2}$, $b_{i,3}$) are called the *associated bands* of the Hopf chord (respectively Borromean chord). An edge e of G (or $f(e)$) is called an *associated edge* of a chord if $f(e)$ has intersection with the chord. The set of the associated edges of a chord C is denoted by $\varepsilon(C)$.

From now on we consider embeddings up to ambient isotopy without explicit mention.

The following Lemma 2.1(1) is folklore in knot theory (cf. [17,22]) and Lemma 2.1(2) is a natural generalization of [23, Lemma].

Lemma 2.1.

(1) *Let* $f, g : G \to S^3$ *be any embeddings. Then* g *is a band sum of Hopf links and* f.

(2) *Let* $f, g : G \to S^3$ *be delta equivalent embeddings. Then* g *is a band sum of Borromean rings and* f.

Proof. Fig. 6 shows that a crossing change (respectively delta move) is equivalent to a band sum of a Hopf link (respectively Borromean ring).

By the assumption there is a finite sequence of embeddings $f = f_0, f_1, \ldots, f_m = g : G \to S^3$ and orientation preserving embeddings $\varphi_1, \ldots, \varphi_m : B^3 \to S^3$ such that f_{i-1} and f_i are related by (D_1, D_2) and φ_i for each $i \in \{1, \ldots, n\}$ where (D_1, D_2) is a crossing change (respectively delta move). Then we have that f_m is a band sum of a Hopf link (respectively Borromean ring) and f_{m-1}. Now we assume inductively that f_m is a band sum of Hopf links (respectively Borromean rings) and f_k. Suppose that the Hopf chords (respectively Borromean chords) have intersection with $\varphi_k(B^3)$. Then we sweep and slide them out of $\varphi_k(B^3)$ up to ambient isotopy of f_m without changing f_k. Note that this deformation is possible since the tangle D_2 is trivial. Thus we have that

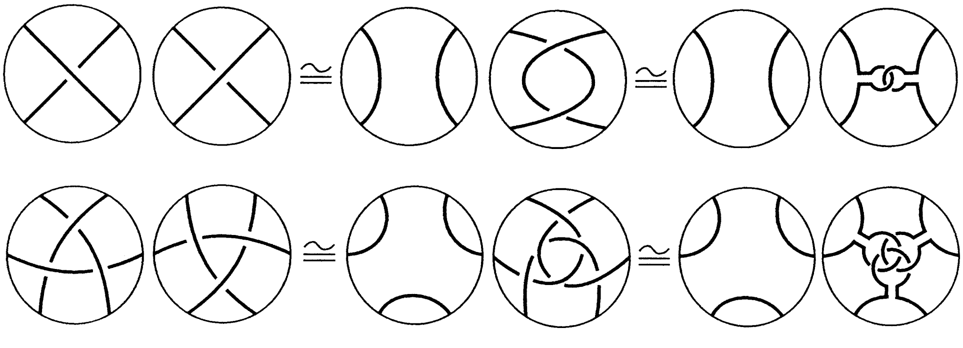


Fig. 6.

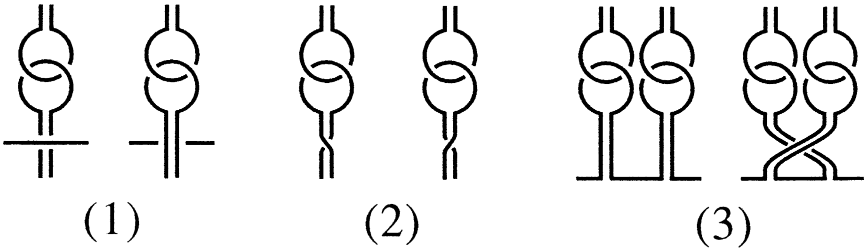


Fig. 7.

f_m is a band sum of Hopf links (respectively Borromean rings) and f_k so that the Hopf chords (respectively Borromean chords) are away from $\varphi_k(B^3)$. Then taking a Hopf link (respectively Borromean ring) and its associated bands in $\psi_k(B^3)$ we have that f_m is a band sum of Hopf links (respectively Borromean rings) and f_{k-1}. In this way we finally have that $g = f_m$ is a band sum of Hopf links (respectively Borromean rings) and $f_0 = f$. This completes the proof. □

Lemma 2.2. *Each pair of the embeddings illustrated in Fig.* 7 *are delta equivalent.*

Proof. See Fig. 8. □

Remark 2.3. By Lemma 2.2(1) we have that a simultaneous change of two subsequent crossings on a string with an associated band of a Hopf link is realized, no matter how far the crossings are from the Hopf link, by an application of a delta move, see Fig. 9.

By two applications of Lemma 2.2(1) we have that a clasp-pass move is realized by delta moves. Namely we have the following corollary.

Corollary 2.4. *Clasp-pass equivalent embeddings are delta equivalent.*

Proof of Theorem 1.3. It is easy to check that linking number does not change under an application of a delta move. Then we have that delta equivalent links have the same

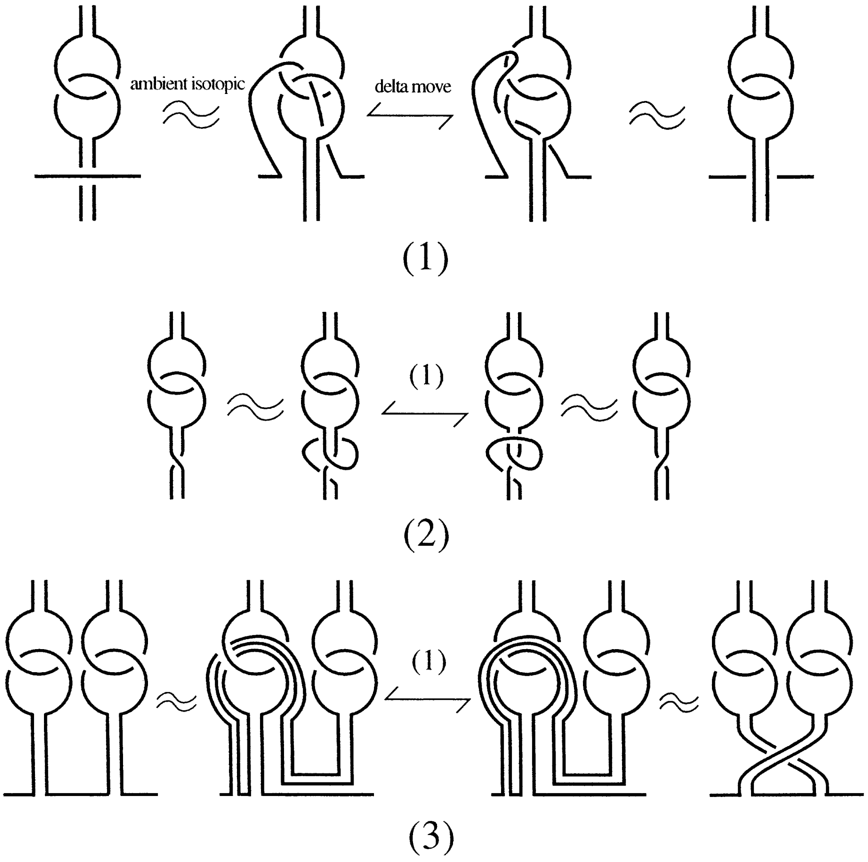


Fig. 8.

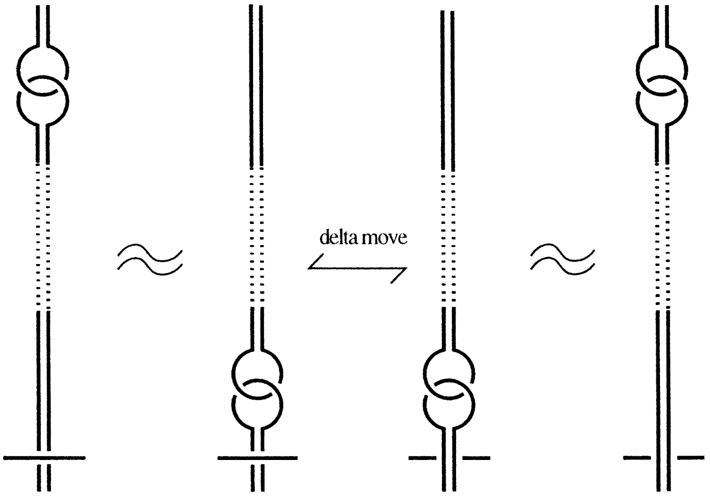
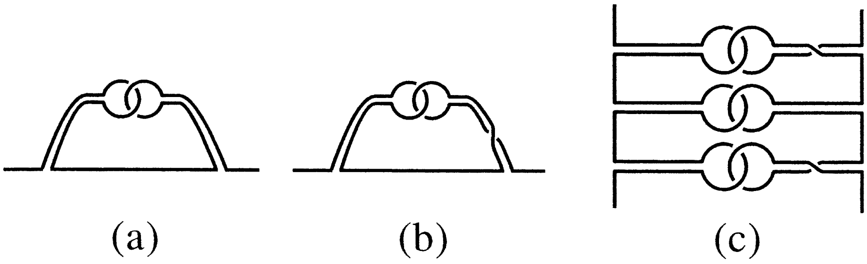


Fig. 9.



(a) (b) (c)

Fig. 10.

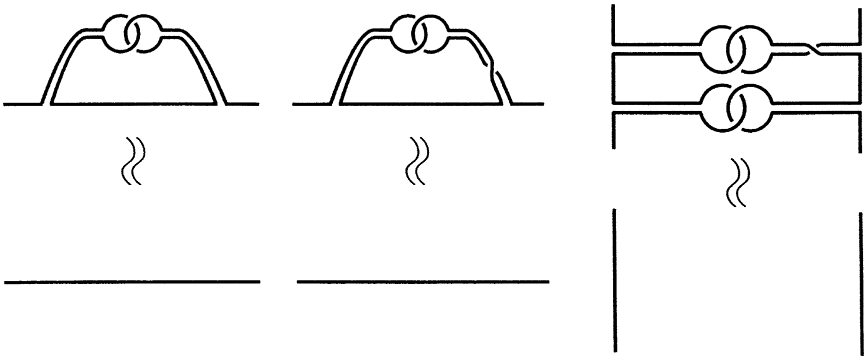


Fig. 11.

linking numbers. We will show the converse. We will deform L up to delta equivalence to a 'canonical form'. Then we will deform M also to a 'canonical form'. It turns out that if L and M have the same linking numbers then these canonical forms are delta-equivalent.

By Lemma 2.1(1) we have that L is a band sum of Hopf links and a trivial link X. Then using Lemma 2.2 we deform L up to delta equivalence so that:

(1) each Hopf chord joining the same component of X is contained in a small 3-ball as illustrated in Fig. 10(a) or (b), and
(2) all Hopf chords joining the same pair of components of X are parallel, see for example Fig. 10(c).

Then by the deformation illustrated in Fig. 11 we finally have that L is delta equivalent to a band sum of Hopf links and a trivial link so that:

(1) no Hopf chord joins the same component, and
(2) all Hopf chords joining the same pair of components are parallel, each of them have no twists of bands or each of them has just a half twist of bands, which depends on the sign of the linking number, and therefore the number of such Hopf chords equals the absolute value of the linking number.

Next we deform M up to delta equivalence to a similar form of band sum of Hopf links and a trivial link. We may suppose that the trivial links are identical. Then by Lemma 2.2 the Hopf chords for L and those for M are transformed into each other by delta moves. Thus L and M are delta equivalent. □

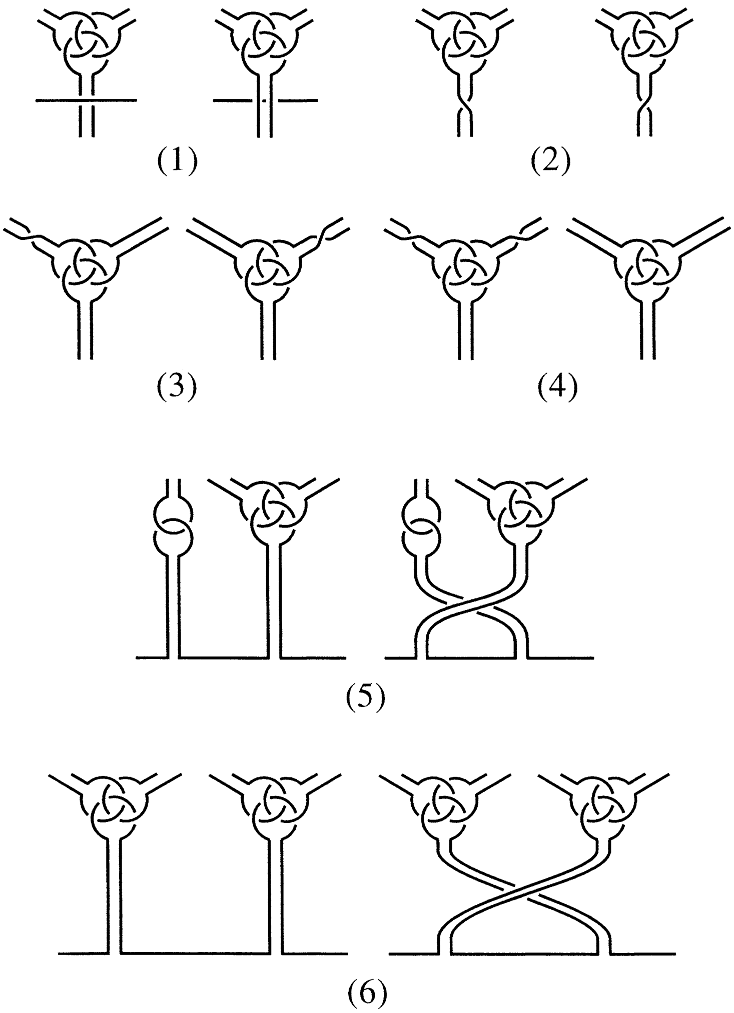
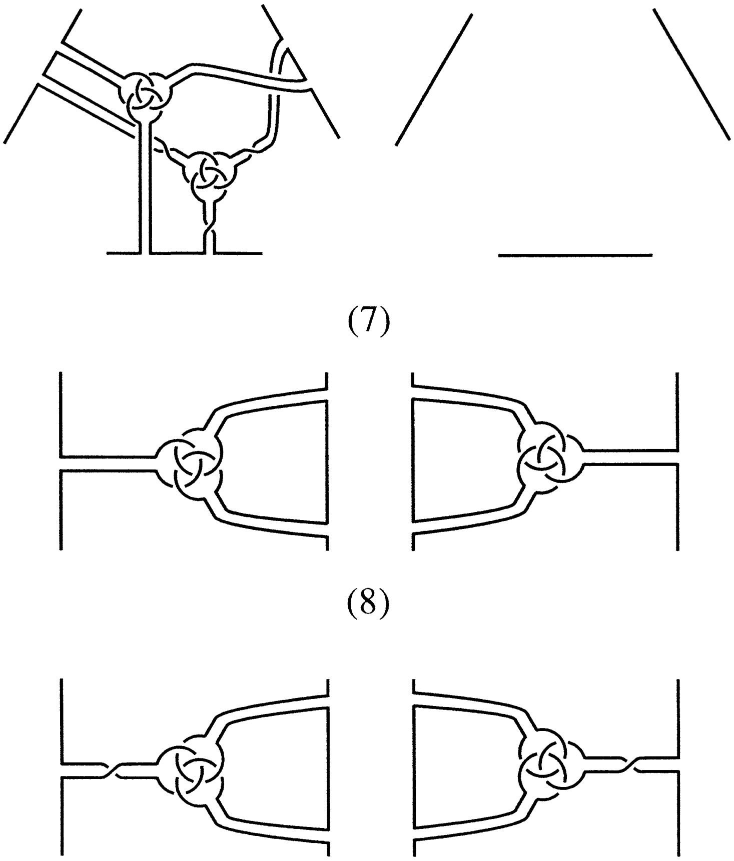


Fig. 12a.

Lemma 2.5. *Each pair of the embeddings illustrated in Fig.* 12 *are clasp-pass equivalent.*

Proof. For (1), see Fig. 13. Then the proofs of (2), (5) and (6) are analogous to that of Lemma 2.2 and we omit them. For (3), see Fig. 14. Then (4) is an immediate consequence of (3). For (7), see Fig. 15. For (8) and (9), it is easy to see that two embeddings are ambient isotopic. □

As before we remark here that a simultaneous change of two subsequent crossings on a string with an associated band of a Borromean ring is realized, no matter how the crossings are far from the Borromean ring, by an application of a clasp-pass move. By this fact and by Lemma 2.5(5) and (6) we have that the positions of the bands attaching to Borromean


(7)

(8)

(9)

Fig. 12b.

rings are freely changeable up to clasp-pass equivalence. By Lemma 2.5 (2), (3), (4) and (7) we have the following. Let $b_{1,1} \cup b_{1,2} \cup b_{1,3} \cup \psi_1(B^3)$ and $b_{2,1} \cup b_{2,2} \cup b_{2,3} \cup \psi_2(B^3)$ be two Borromean chords such that $b_{1,p}$ is next to $b_{2,p}$ on $f(G)$ for each $p = 1, 2, 3$ and the total number of half-twists of bands differs by an odd number. Then they cancel up to clasp-pass equivalence. In the following proofs of theorems we use these facts without explicit mention.

All strategies of the proofs below are essentially the same as that of Theorem 1.3. Namely after showing that certain invariants such as a_2 etc. are clasp-pass equivalence invariants, we deform a link or a spatial graph to a 'canonical form'. It then turns out that canonical forms with the same invariants are clasp-pass equivalent.

Lemma 2.6. *Let H be a subgraph of a graph G. Let $f, g : G \to S^3$ be clasp-pass equivalent embeddings. Then the restriction maps $f|_H, g|_H : H \to S^3$ are clasp-pass equivalent.*

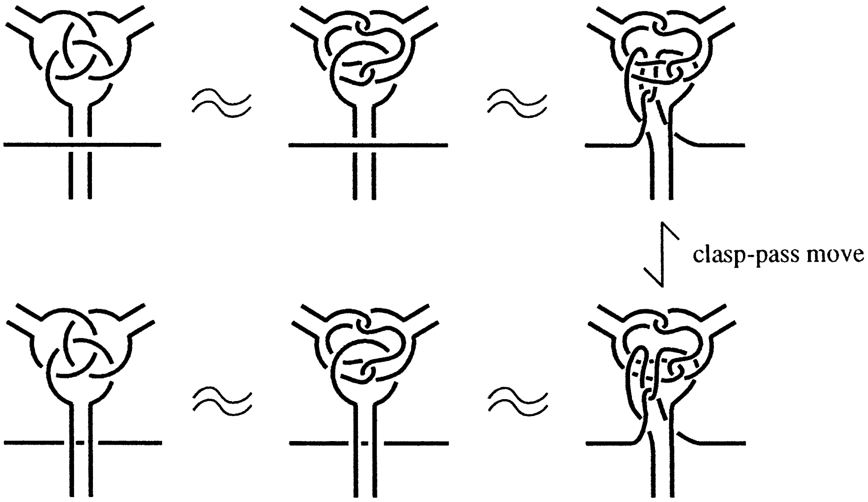


Fig. 13.

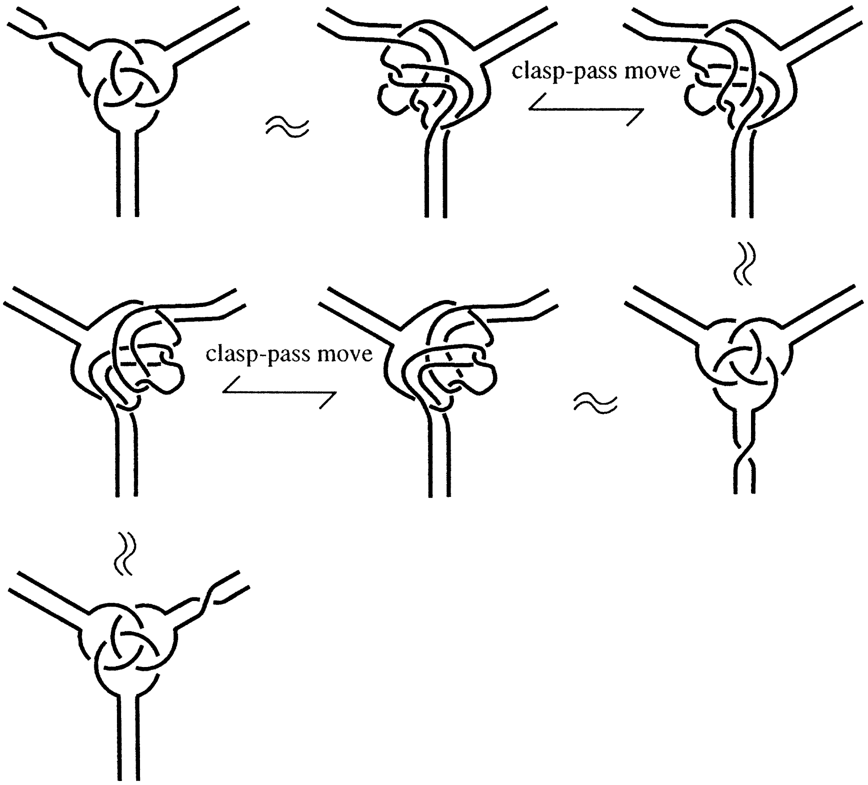


Fig. 14.

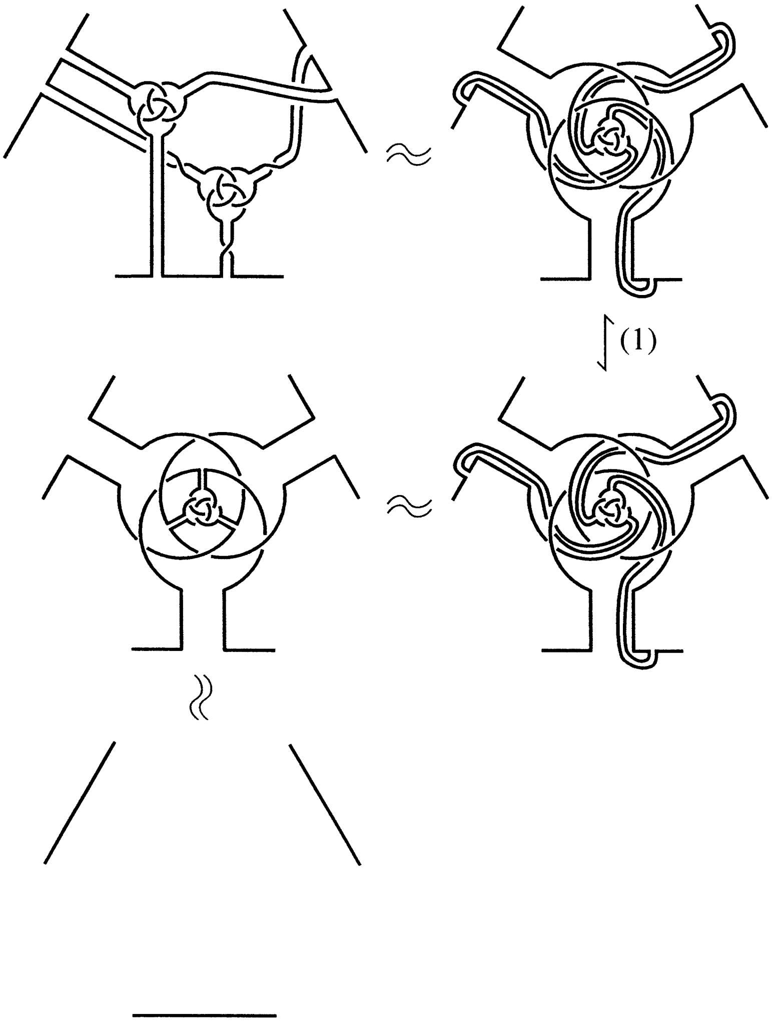


Fig. 15.

Proof. It is sufficient to consider the case that f and g are related by single clasp-pass move. If the four strings that appear in the clasp-pass move are all contained in the image of H then $f|_H$ and $g|_H$ are also related by the same clasp-pass move. If not then we have $f|_H$ and $g|_H$ are ambient isotopic. This completes the proof. □

Lemma 2.7.

(1) *If oriented knots L and M are clasp-pass equivalent then $a_2(L) = a_2(M)$.*

(2) *If n-component oriented links L and M are clasp-pass equivalent then $a_{n+1}(L) \equiv a_{n+1}(M) \pmod 2$.*

(3) *If three-component ordered oriented links L and M are clasp-pass equivalent then $\mu(L) \equiv \mu(M)$ modulo the linking numbers of two-component sublinks.*

Proof. (1) This is an easy half of Theorem 1.2. In fact it easily follows from the formula $a_2(K_+) - a_2(K_-) = \ell k(L_0)$ where K_- is an oriented knot obtained from an oriented knot K_+ by changing a positive crossing to a negative one, and L_0 is an oriented 2-component link obtained by smoothing the crossing [6].

(2) Note that a clasp-pass move is realized by a pass move. Pass moves preserve both the linking number and the Arf invariant [11], and so do clasp-pass moves. Lemma 2.6 and [21, Theorem] complete the proof.

(3) Since μ is a *link homotopy* invariant [9] it is sufficient to show that clasp-pass equivalence implies link homotopy for three-component links. Note that each of the tangles of clasp-pass move contains four strings. Since L is a three-component link at least two of them belong to the same component. Then it is easy to check that a clasp-pass move is realized by link homotopy. □

Proof of Theorem 1.4. It follows from [21, Theorem] that the conditions (2) and (3) are equivalent. By Lemmas 2.6 and 2.7 we have that (1) implies (2). We show the converse. Since L is algebraically split we have by Theorem 1.3 that L is delta equivalent to an n-component trivial link $X = Y_1 \cup \cdots \cup Y_n$. Then by Lemma 2.1(2) we have that L is a band sum of Borromean rings and X. Let C be a Borromean chord of the band sum. We say that the *type* of C is (i, j, k) if $\varepsilon(C) = \{Y_i, Y_j, Y_k\}$, (i, j) if $\varepsilon(C) = \{Y_i, Y_j\}$ and (i) if $\varepsilon(C) = \{Y_i\}$. Using Lemma 2.5 we deform L up to clasp-pass equivalence so that:

(1) each Borromean chord of type (i) is contained in a 3-ball as illustrated in Fig. 16(a) or (b), and for each i, not both of (a) and (b) occur,

(2) each Borromean chord of type (i, j) is contained in a 3-ball as illustrated in Fig. 16(c) or (d), and

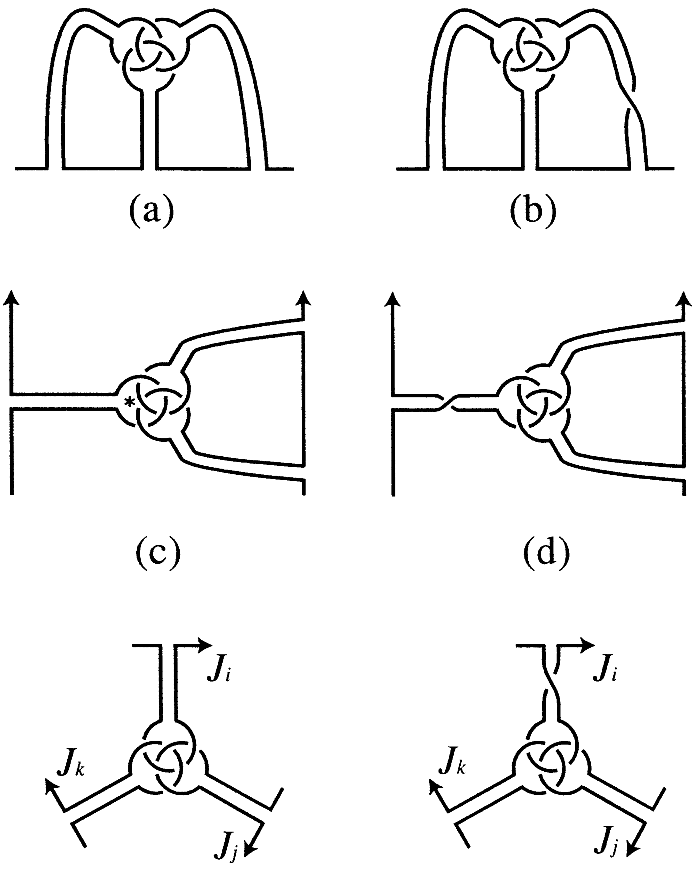
(3) each Borromean chord of type (i, j, k) is contained in a 3-ball as illustrated in Fig. 16(e) or (f), and for each i, j, k, not both of (e) and (f) occur.

By sliding one of the three bands of a Borromean chord of type (i, j) along Y_i as illustrated in Fig. 17 and by Lemma 2.5 we have that two Borromean chords as illustrated in Fig. 16(c) and (d) are transformed into each other by clasp-pass moves. Then by Lemma 2.5 we have that two Borromean chords of type (i, j) cancel each other.

Therefore we have:

(4) for each $1 \leqslant i < j \leqslant n$, there is at most one Borromean chord of type (i, j) and if there is, it is contained in a 3-ball as illustrated in Fig. 16(c).

Then we deform M to a similar form and compare them. Note that the local knots illustrated in Fig. 16(a) and (b) are a trefoil knot and a figure eight knot, respectively. Since $a_2(\text{trefoil knot}) = 1$, $a_2(\text{figure eight knot}) = -1$ and a_2 is additive under connected sum of knots [6], we may suppose, by the condition $a_2(J_i) = a_2(K_i)$ and by the invariance of a_2 under clasp-pass equivalence, that the Borromean chords of type (i) for L and those for M are identical. Using the skein relation at the marked crossing point in Fig. 16(c) together with the result in [4] we can check that $a_3(J_i \cup J_j) \equiv 0 \pmod 2$ if and only if there are even number of Borromean chords of type (i, j). By a calculation we have that $\mu(J_i \cup J_j \cup J_k)$



(a) (b)

(c) (d)

(e) (f)

Fig. 16.

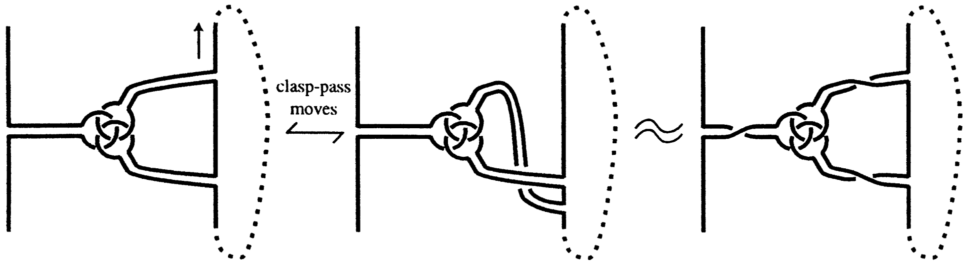


Fig. 17.

is equals to the signed number of Borromean chords of type (i, j, k) where the sign is 1 if they are as illustrated in Fig. 16(e) and -1 if they are as illustrated in Fig. 16(f). Thus we have the result. □

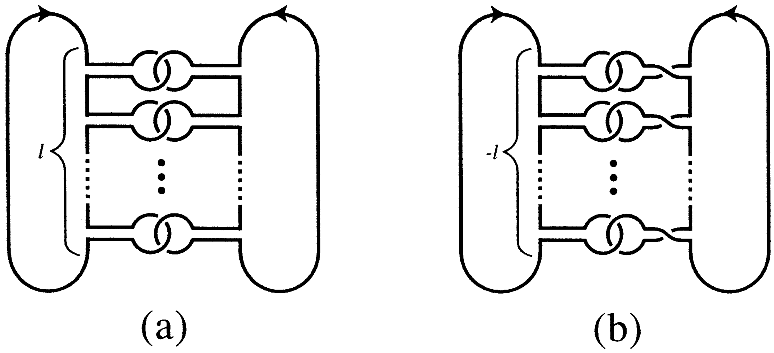


Fig. 18.

Proof of Theorem 1.5. It follows from [21, Theorem] that the conditions (2) and (3) are equivalent. By Lemmas 2.6 and 2.7 we have that (1) implies (2). We show the converse. Let $\ell k(J_1, J_2) = l$. Then by Theorem 1.3 we have that L is delta equivalent to a 2-component link $X_l = Y_1 \cup Y_2$ as illustrated in Fig. 18(a) or (b) according to l is positive or negative. Then by Lemma 2.1(2) we have that L is a band sum of Borromean rings and X_l.

Then as in the proof of Theorem 1.4 we deform L up to clasp-pass equivalence so that:

(1) each Borromean chord of type (i) is contained in a 3-ball as illustrated in Fig. 16(a) or (b), and for each i, not both of (a) and (b) occur,

(2) there is at most one Borromean chord of type $(1, 2)$ and if there is, it is contained in a 3-ball as illustrated in Fig. 16(c).

Suppose that l is even. Then as in the proof of Theorem 1.4 we can check that the parity of a_3 changes after attaching a Borromean chord of type $(1, 2)$. However when l is odd the parity does not change. Therefore we must show that when l is odd we can erase a Borromean chord of type $(1, 2)$ by clasp-pass moves. First we give a full-twist to X_l and perform $l(l-1)/2$ applications of a clasp-pass move. Then we perform l applications of a delta-move by Lemma 2.2(2) and back to the original form of X_l as illustrated in Fig. 19. We replace each of the delta moves above by band sum of Borromean rings. By the proof of Lemma 2.2 we have that the type of each Borromean chord is $(1, 2)$. Thus we have that X_l is clasp-pass equivalent to X_l together with l Borromean chords of type $(1, 2)$. If we consider the clasp-pass equivalence above together with the Borromean chords for L, then we have, with some additional clasp-pass moves, that L is clasp-pass equivalent to a band sum of L and l Borromean chords of type $(1, 2)$. Since l is odd we have the desired conclusion. □

Proof of Theorem 1.7. It follows from [21, Theorem] that the conditions (2) and (3) are equivalent. By Lemmas 2.6 and 2.7 we have that (1) implies (2). We show the converse. Let $\ell k(J_1, J_2) = l_1$, $\ell k(J_2, J_3) = l_2$ and $\ell k(J_3, J_1) = l_3$. Then by Theorem 1.3 we have that L is delta equivalent to a 3-component link $X_{l_1,l_2,l_3} = Y_1 \cup Y_2 \cup Y_3$ as illustrated in Fig. 20 where the case $l_1 = 2$, $l_2 = -2$, $l_3 = 3$ is illustrated.

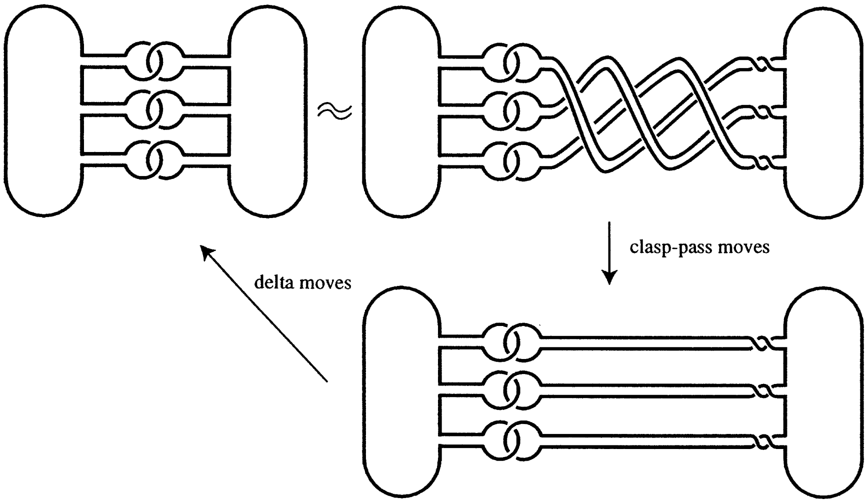


Fig. 19.

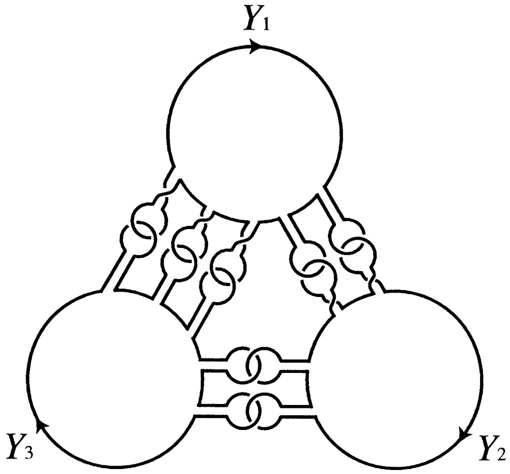


Fig. 20.

Then by Lemma 2.1(2) we have that L is a band sum of Borromean rings and X_{l_1,l_2,l_3}. Then as in the proofs of Theorems 1.4 and 1.5 we deform L up to clasp-pass equivalence so that:

1. each Borromean chord of type (i) is contained in a 3-ball as illustrated in Fig. 16(a) or (b), and for each i, not both of (a) and (b) occur,
2. for each i there is at most one Borromean chord of type $(i, i+1)$ (here we consider $3+1=1$) and if there is, it is contained in a 3-ball as illustrated in Fig. 16(c), and
3. each Borromean chord of type $(1,2,3)$ is contained in a 3-ball as illustrated in Fig. 16(e) or (f), and not both of (e) and (f) occur.

Then we deform M up to clasp-pass equivalence to a band sum of Borromean rings and X_{l_1,l_2,l_3} satisfying (1), (2) and (3).

As before we have that the Borromean chords of type (i) for L and for M correspond to $a_2(J_i)$ and $a_2(K_i)$, respectively. Since $a_2(J_i) = a_2(K_i)$ we may suppose that they are identical. Next we consider Borromean chords of type $(1,2,3)$. Note that $\mu(L)$ is determined up to the greatest common divisor d of l_1, l_2 and l_3. By a calculation we have that the signed number of Borromean chords of type $(1,2,3)$ for L (respectively M) is congruent to $\mu(L)$ (respectively $\mu(M)$) modulo d. By the deformation of X_{l_1,l_2,l_3} as illustrated in Fig. 21 we have that L is clasp-pass equivalent to a band sum of l_i Borromean rings of type $(1,2,3)$, l_i Borromean rings of type $(i,i+1)$ and L. Thus we have that the

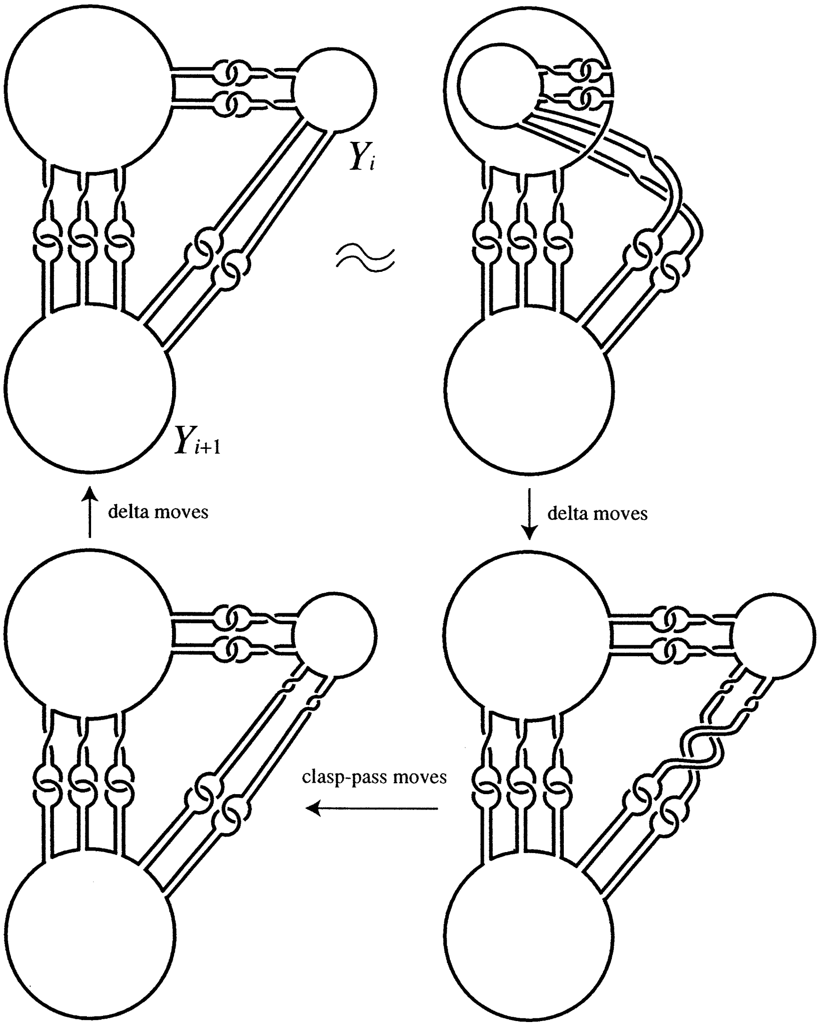


Fig. 21.

number of Borromean chords of type $(1, 2, 3)$ is changeable up to each l_i, hence up to d. Therefore we may suppose that the Borromean chords of type $(1, 2, 3)$ for L and those for M are identical. Note that by further applications of a clasp-pass move we have that the conditions (1), (2) and (3) still hold. Next we will deform L up to clasp-pass equivalence so that the Borromean chords of type $(i, i + 1)$ for L coincides with those for M. Suppose that all l_i are even. Then by the conditions $a_3(J_i \cup J_{i+1}) \equiv a_3(K_i \cup K_{i+1}) \pmod 2$ we have the result as in the proof of Theorem 1.5. Next suppose that one or two of l_1, l_2 and l_3 are odd. Suppose that l_i is odd. Then by the deformation illustrated in Fig. 22 and Lemma 2.5 we can create or eliminate a Borromean chord of type $(i, i + 1)$. Thus we have the result. Finally suppose that $l_1l_2l_3$ is odd. Then by the deformation illustrated in Fig. 22 and Lemma 2.5 we can replace a Borromean chord of type $(i, i + 1)$ by a Borromean chord of type $(i + 1, i + 2)$. However the parity of the total number of Borromean chords of types $(1, 2)$, $(2, 3)$ and $(3, 1)$ is invariant under this deformation. By a calculation using the skein relation at the marked crossing point in Fig. 16(c) we have that the parity of $a_4(L)$ changes under a band sum of a Borromean chord of type $(i, i + 1)$. Therefore we have that the parity for L is equal to that for M. Thus we have the result. □

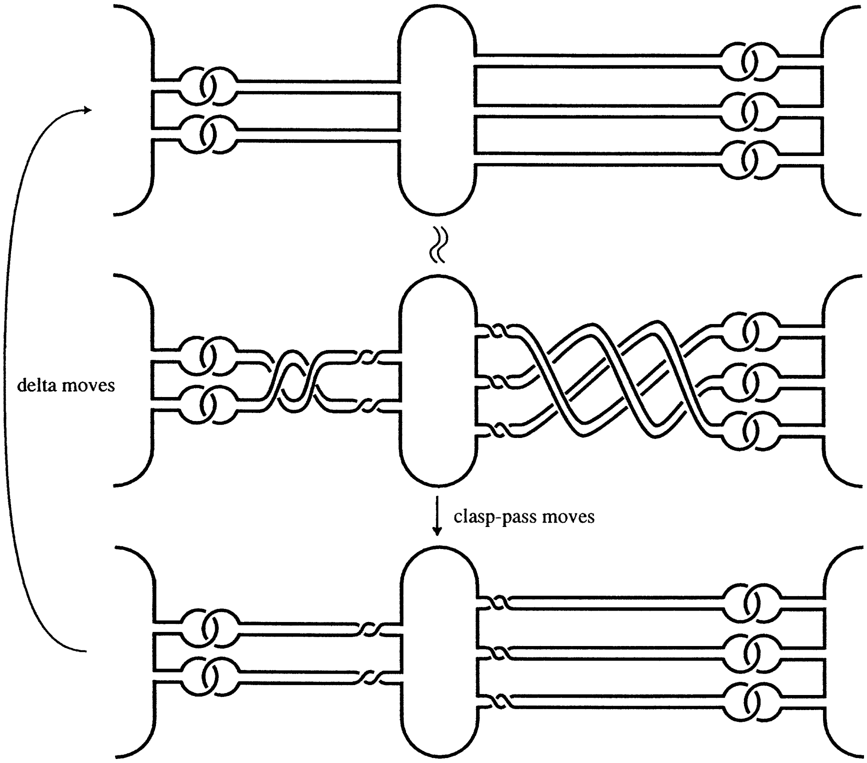


Fig. 22.

3. Clasp-pass moves on spatial graphs

In this section we give a proof of Theorem 1.11. We use the following characterization of planar graphs without disjoint cycles.

Let G be a finite graph. We denote the set of all edges of G by $E(G)$ and the set of all vertices of G by $V(G)$. Let W be a subset of $V(G)$. Then $G - W$ denotes the maximal subgraph of G with $V(G - W) = V(G) - W$. A finite graph G is called a *generalized bouquet* if there is a vertex v of G such that $G - \{v\}$ does not contain cycles. A *cycle* is a graph that is homeomorphic to a circle. A loopless graph G is called a *multi-spoke wheel* if there is a vertex v of G such that $G - \{v\}$ is a cycle. Then the edges incident to v are called *spokes* and the edges that are not spokes are called *tires*. A *double trident* is a planar graph as illustrated in Fig. 23. Here the marked edges are allowed to have multiplicity. Namely there may be more edges joining the same vertices.

Let G be a finite graph. The *reduced graph of G* is the maximal subgraph of G without vertices of degree less than two.

Theorem 3.1 [18]. *Let G be a planar graph without disjoint cycles. Then at least one of the followings holds*:

(1) *G is a generalized bouquet,*
(2) *the reduced graph of G is homeomorphic to a multi-spoke wheel,*
(3) *the reduced graph of G is homeomorphic to a subgraph of a double trident.*

We also need, as a sequel to Lemma 2.5, the following Lemma 3.2.

Lemma 3.2. *Each pair of the embeddings illustrated in Fig.* 24 *are clasp-pass equivalent.*

Proof. (1) It is easy to see that these two embeddings are ambient isotopic.

(2) By (1) we add one more Borromean chord to the left-hand side. Then we deform the two Borromean chords using Lemma 2.5 so that they cancel each other by Lemma 2.5(7).

(3) The proof is illustrated in Fig. 25. □

Proof of Theorem 1.11. The ‘only if’ part follows by Lemmas 2.6 and 2.7. We show the ‘if’ part. Let $u : G \to S^3$ be an embedding such that there is a 2-sphere $S \subset S^3$ with

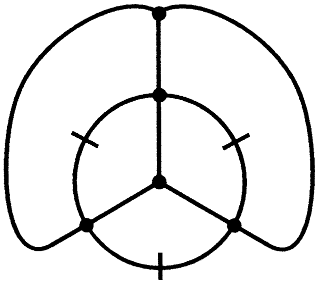
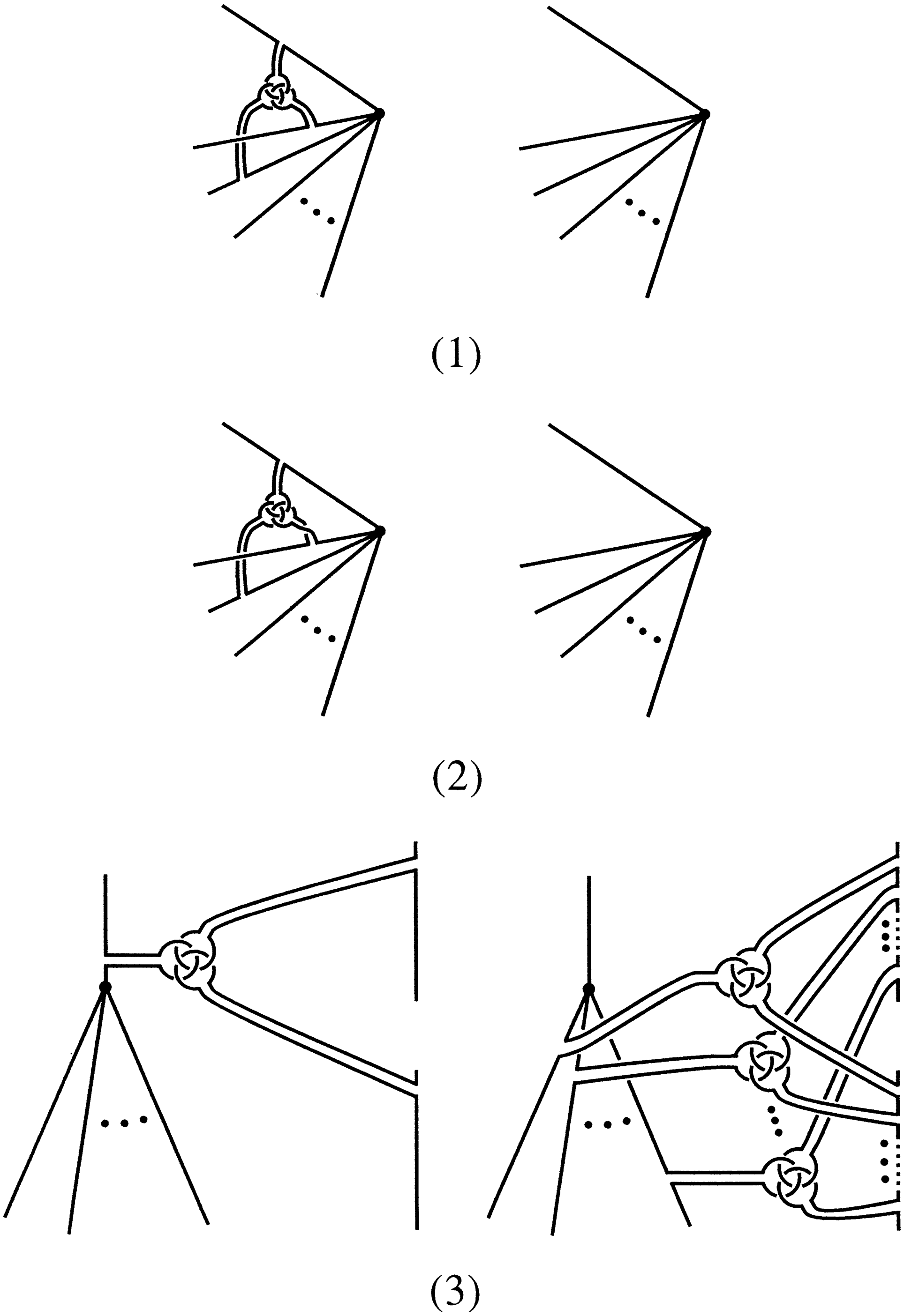


Fig. 23.



(1)

(2)

(3)

Fig. 24.

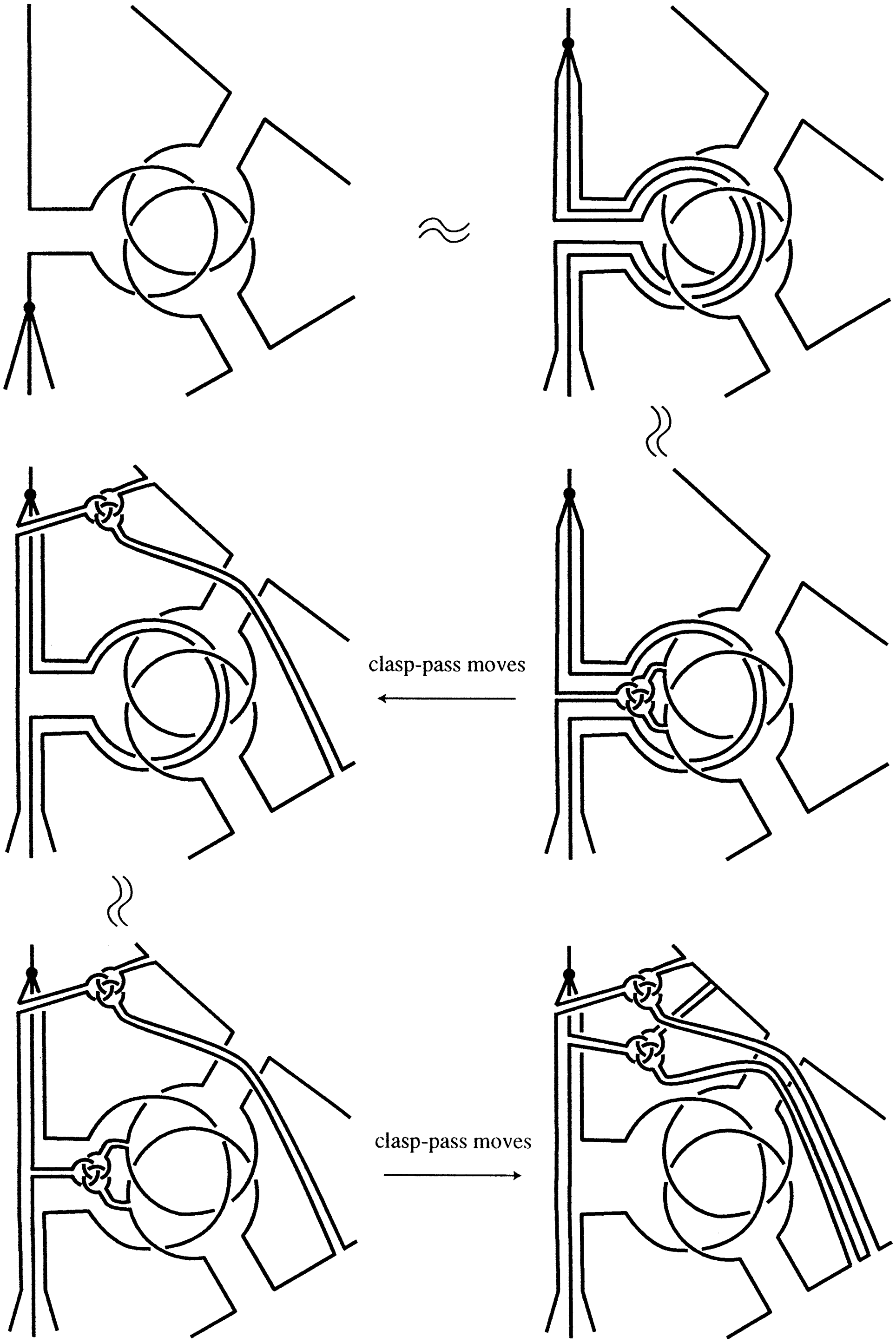


Fig. 25.

$u(G) \subset S$. Note that such an embedding is unique up to ambient isotopy in S^3 [8]. Let $f : G \to S^3$ be an embedding. Then by Theorem 1.10 and Lemma 2.1(2) we have that f is a band sum of Borromean rings and u. Using Lemmas 2.5 and 3.2 we will deform f up to clasp-pass equivalence to a special form of band sum of Borromean rings and u. Similarly we will deform g to a special form of band sum of Borromean rings and u. Then we will compare them and using the assumption we will show that they are clasp-pass equivalent. We note that when we use Lemma 3.2(3) we always apply it in the direction that increases the number of the Borromean chords. Namely "by an application of Lemma 3.2(3) at a vertex v" we mean the change from left-hand side embedding of Fig. 24(3) to the right-hand side one where the vertex in Fig. 24(3) is considered as v. According to Theorem 3.1 we divide the proof into the following three cases.

Case 1. G is a generalized bouquet.

By subdividing G if necessary, we may assume that G has no loops. Let v be a vertex of G such that $G - \{v\}$ does not contain cycles. Suppose that f is a band sum of Borromean rings and u. We will deform f up to clasp-pass equivalence. By repeated applications of Lemma 3.2(3) we have that all bands are attached to the edges incident to v since $G - \{v\}$ does not contain cycles. Then by Lemma 3.2(1) and (2) we erase each Borromean chord whose associated edges are three different edges. Then by applications of Lemma 3.2(3) at v we replace each Borromean chord which has only one associated edge by the Borromean chords each of which has just two associated edges. Thus we have that the associated edges of each Borromean chord are just two edges incident to v. Suppose that the two edges are not on any cycle of G. Then by deforming u up to ambient isotopy if necessary, we have that the two edges are next to each other on the 2-sphere S containing $u(G)$. Then we erase the Borromean chord by the deformation illustrated in Fig. 26 and Lemma 2.5. Then we deform g up to clasp-pass equivalence to a similar form of a band sum of Borromean rings and u. Note that for each Borromean chord there is a unique cycle γ of G containing the two associated edges. By the assumption and by the invariance of a_2 under clasp-pass moves we have that $a_2(f(\gamma)) = a_2(g(\gamma))$. Then by Lemma 2.5 we have that the Borromean chords of f and g with respect to the cycle γ are transformed into each other by clasp-pass moves. Thus we have that f and g are clasp-pass equivalent.

Case 2. The reduced graph of G is homeomorphic to a multi-spoke wheel.

We may suppose without loss of generality that G itself is a multi-spoke wheel. Let v be the vertex of G such that $G - \{v\}$ is a cycle. Let $v_1, \dots, v_n$ be the vertices of the cycle in a cyclic order. Let $e_{i,1}, e_{i,2}, \dots$ be the edges of G joining v and v_i. Let d_i be the edge joining v_i and v_{i+1} where $n + 1 = 1$.

We will deform f up to clasp-pass equivalence to a special form of a band sum of Borromean rings and u such that each Borromean chord R satisfies one of the followings:

(a) $\varepsilon(R) = \{d_1\}$,

(b) $\varepsilon(R) = \{e_{i,j}, e_{i,k}\}$ for some i, j, k with $j \neq k$,

(c) $\varepsilon(R) = \{d_i, e_{j,k}, e_{l,m}\}$ for some i, j, k, l, m with $j \neq l$.

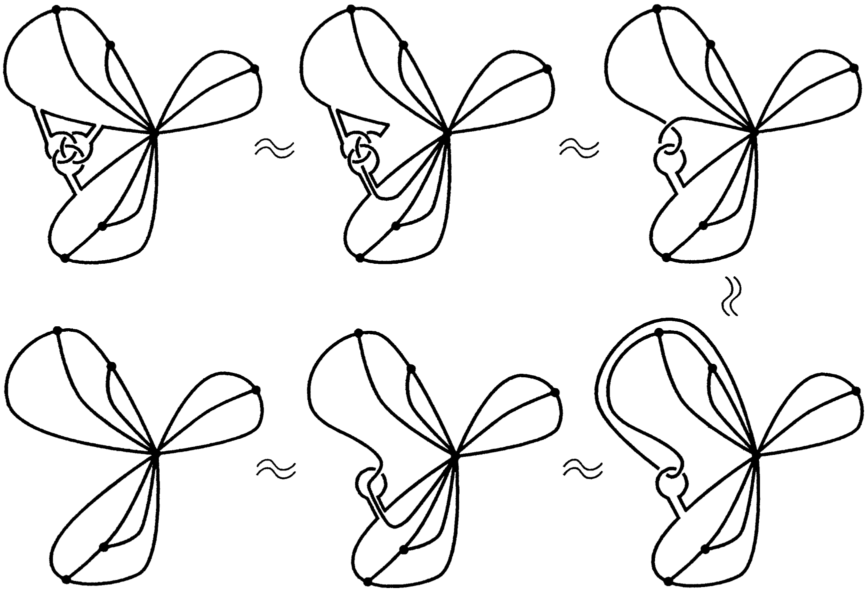


Fig. 26.

Suppose that f is a band sum of Borromean rings and u. We will deform f up to clasp-pass equivalence to the form above step by step as follows. Each step will be done without disturbing the preceding steps.

Step 1. Erase each Borromean chord R with $\varepsilon(R) = \{e\}$ for some edge e of G with $e \neq d_1$.

This is possible by Lemma 3.2(3) and by the fact that G has no loops.

Step 2. Erase each Borromean chord whose set of the associated edges contains only tires and not equal to $\{d_1\}$.

Using Lemma 3.2(3) we slide the bands along the cycle of the tires toward d_1 and we have the result.

Step 3. Erase each Borromean chord whose associated edges are one spoke and two tires, or one spoke and one tire.

For both cases, by using Lemma 2.5(8) or (9) if necessary, we may suppose that one band attaches to a spoke and the other bands attach to tires. Then using Lemma 3.2(3) we slide the two bands attaching to the tires or a tire along the cycle of the tires in mutually opposite directions to the two tires adjacent to the spoke and then we eliminate it by Lemma 3.2(1) or (2).

Step 4. Erase each Borromean chord whose associated edges are two different spokes and one tire that is not the form of (c). Namely the two spokes are mutually multiple edges.

Using Lemma 3.2(3) we slide the band attaching to a tire along the cycle of the tires until it comes to a tire incident to the two spokes and then we eliminate it by Lemma 3.2(1) or (2).

Step 5. Erase each Borromean chord whose associated edges are spokes only that is not the form of (b).

Suppose that the associated edges are $e_{i,j}$ and $e_{k,l}$ with $i \neq k$. Suppose that two bands are attaching to $e_{i,j}$. Then we apply Lemma 3.2(3) at the vertex v_i. Suppose that the associated edges are three different spokes then we eliminate the Borromean chord by Lemma 3.2(1) or (2).

Thus we have that f is a band sum of Borromean rings and u such that each Borromean chord R satisfies one of (a), (b) and (c). Then using Lemma 3.2(3) and Lemma 3.2(1) or (2) we further deform each Borromean chord satisfying (c) to a Borromean chord R satisfying the following condition:

(d) $\varepsilon(R) = \{d_j, e_{j,k}, e_{l,m}\}$ for some j, k, l, m with $j \neq l$.

Next we deform g up to clasp-pass equivalence to a similar form. Namely we have that g is a band sum of Borromean rings and u such that each Borromean chord R satisfies one of (a), (b) and (d).

Then by Lemma 2.5 we perform possible cancellations of the pairs of the Borromean chords of different parity of the total number of half-twists of bands for each of f and g. Now we are ready to compare f and g.

Note that the Borromean chords satisfying (b) or (d) do not affect the knot type of the cycle $G - \{v\}$. Therefore by the condition $a_2(f(G - \{v\})) = a_2(g(G - \{v\}))$ we have that the number of the Borromean chords of f satisfying (a) is equal to that of g. Thus they are transformed into each other by clasp-pass moves. Next we consider the Borromean chords satisfying (d) for each j, k, l, m with $j \neq l$. Note that there is just one cycle of G containing the edges d_j, $e_{j,k}$ and $e_{l,m}$. Note that the Borromean chords satisfying (b) do not affect the knot type of this cycle. Therefore we have that the number of the Borromean chords of f satisfying (d) for j, k, l, m with $j \neq l$ is equal to that of g. Thus they are also transformed into each other by clasp-pass moves. Finally we consider the 2-cycle of the edges $e_{i,j}$ and $e_{i,k}$. Using Lemma 2.5 we have that the Borromean rings attaching to $e_{i,j}$ and $e_{i,k}$ of f and g are transformed into each other by clasp-pass moves. Thus we have that f and g are clasp-pass equivalent.

Case 3. The reduced graph of G is homeomorphic to a subgraph of a double trident.

We will show the case that G itself is a double trident. Other cases are essentially the same and we omit them. We name the edges of G as illustrated in Fig. 27.

We consider the indices of the edges modulo 3. Namely we consider that $e_{3+1} = e_1$, $d_{3+2} = d_2$, $e_{3+1,i} = e_{1,i}$ etc. We will deform f up to clasp-pass equivalence to a band sum of Borromean rings and u such that each Borromean chord R satisfies one of the following conditions:

(a) $\varepsilon(R) = \{e_{1,i}, e_{2,j}, e_{3,k}\}$ for some i, j, k,
(b) $\varepsilon(R) = \{e_{i,j}, e_{i+1,k}, e_{i+2}\}$ for some i, j, k,
(c) $\varepsilon(R) = \{e_{i,j}, e_{i+1,k}, d_i\}$ for some i, j, k,
(d) $\varepsilon(R) = \{e_{i,j}, e_{i,k}\}$ for some i, j, k with $j \neq k$,
(e) $\varepsilon(R) = \{e_{i,j}, e_i, e_{i+1}\}$ for some i, j,
(f) $\varepsilon(R) = \{e_{i,j}, d_i, d_{i+1}\}$ for some i, j,

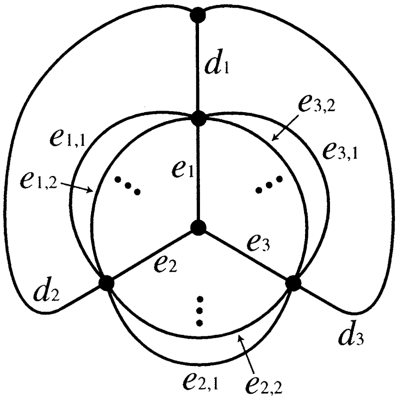


Fig. 27.

(g) $\varepsilon(R) = \{e_{i,j}, e_i, d_{i+1}\}$ for some i, j,
(h) $\varepsilon(R) = \{e_{i,j}, d_i, e_{i+1}\}$ for some i, j,
(i) $\varepsilon(R) = \{e_i, d_{i+2}\}$ for some i.

For this purpose we will deform f step by step as in case 2 without disturbing the previous steps.

Step 1. Erase each Borromean chord R with $\varepsilon(R) = \{e\}$ for some edge e of G.

Step 2. Erase each Borromean chord R with $\varepsilon(R) = \{e_i, d_{i+1}, d_{i+2}\}$ or $\{d_i, e_{i+1}, e_{i+2}\}$ for some i.

Step 3. Erase each Borromean chord R with $\varepsilon(R) = \{e_i, e_{i+1}, d_i\}$, $\{e_i, e_{i+1}, d_{i+1}\}$, $\{d_i, d_{i+1}, e_i\}$ or $\{d_i, d_{i+1}, e_{i+1}\}$ for some i.

Step 4. Erase each Borromean chord R with $\varepsilon(R) = \{e_i, d_i\}$ for some i.

Step 5. Erase each Borromean chord R with $\varepsilon(R) = \{e_i, d_{i+1}\}$ for some i.

Step 6. Erase each Borromean chord R with $\varepsilon(R) = \{e_i, e_{i+1}\}$ or $\{d_i, d_{i+1}\}$ for some i, j.

Step 7. Erase each Borromean chord R with $\varepsilon(R) = \{e_i, d_i, e_{i+1,j}\}$ for some i, j.

Step 8. Erase each Borromean chord R with $\varepsilon(R) = \{e_i, d_{i+1}, e_{i+1,j}\}$, $\{e_i, d_{i+2}, e_{i+1,j}\}$, $\{d_i, e_{i+1}, e_{i+1,j}\}$ or $\{d_i, e_{i+2}, e_{i+1,j}\}$ for some i, j.

Step 9. Erase each Borromean chord R with $\varepsilon(R) = \{e_i, e_{i+1}, e_{i+1,j}\}$, $\{e_i, e_{i+1}, e_{i+2,j}\}$, $\{d_i, d_{i+1}, e_{i+1,j}\}$ or $\{d_i, d_{i+1}, e_{i+2,j}\}$ for some i, j.

Step 10. Erase each Borromean chord R with $\varepsilon(R) = \{e_i, e_{i+1,j}, e_{i+1,k}\}$ or $\{d_i, e_{i+1,j}, e_{i+1,k}\}$ for some i, j, k with $j \neq k$.

Step 11. Erase each Borromean chord R with $\varepsilon(R) = \{e_i, e_{i+1,j}\}$ or $\{d_i, e_{i+1,j}\}$ for some i, j.

Step 12. Erase each Borromean chord R with $\varepsilon(R) = \{e_i, e_{i,j}\}$, $\{e_i, e_{i+2,j}\}$, $\{d_i, e_{i,j}\}$ or $\{d_i, e_{i+2,j}\}$ for some i, j.

Step 13. Erase each Borromean chord R with $\varepsilon(R) = \{e_{i,j}, e_{i+1,k}\}$ for some i, j, k.

Step 14. Erase each Borromean chord R with $\varepsilon(R) = \{e_{i,j}, e_{i+1,k}, e_i\}$ or $\{e_{i,j}, e_{i+1,k}, d_{i+2}\}$ for some i, j, k.

We note that each step above is done by applications of Lemmas 2.5 and 3.2. We omit the details. Then we erase each Borromean chord whose associated edges are three different edges incident to a common vertex by Lemma 3.2(1) or (2).

Next we deform g to a similar form. Now we are ready to compare f and g. First we consider the knot types of the 4-cycle with the edges $e_i, e_{i+1}, d_{i+1}, d_i$. Only the Borromean chords satisfying the condition (i) affect on the knot type of this cycle. Therefore we can control the Borromean chords satisfying (i). Then by considering appropriate cycles we can control other Borromean chords and we have the result. We omit the details. □

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