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ON HIGHER DIMENSIONAL θ-CURVES

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Abstract

In [2], Kinoshita showed that for any collection of $s(s-1)/2$ knot types in R^3, there exists a θ_s-curve in R^3 whose constituent knots correspond to the given $s(s-1)/2$ knot types. In this paper, we define a higher dimensional θ_s-curve, and prove that for any collection of $s(s-1)/2$ ribbon n-knot types, there exists an n-dimensional θ_s-curve in R^{n+2} whose constituent knots correspond to the given $s(s-1)/2$ ribbon n-knot types.

§1. Definitions and Notation

Throughout this paper we work in the piecewise linear category, and we will use the following notation and symbols:

R: the set of real numbers.

$R^n = \{(x_1, \ldots, x_n) \mid x_i \in R\}$: the Euclidean space.

$I = \{x \in R \mid 0 \leq x \leq 1\}$: the closed unit interval.

$D^n = \{(x_1, \ldots, x_n) \in R^n \mid x_1^2 + \cdots + x_n^2 \leq 1\}$.

$D_t^n = \{(x_1, \ldots, x_n) \in R^n \mid x_1^2 + \cdots + x_n^2 \leq t\}$.

An *n-knot* K^n means a locally flat n-sphere in R^{n+2}. An n-knot is *unknotted* if it bounds an $(n+1)$-disk in R^{n+2}. Two knots K_1^n and K_2^n are said to be of the *same knot type*, if there exists an ambient isotopy $\{f_t\}_{t\in I}$ of R^{n+2} such that $f_1(K_1^n) = K_2^n$.

DEFINITION 1.1. (ribbon n-knot, cf. Yanagawa [4]) Let $B_0^{n+1}, B_1^{n+1}, \ldots, B_r^{n+1}$ be mutually disjoint $(n+1)$-disks in R^{n+2} and $\partial B_i^{n+1} = S_i^n$ for $i = 0, 1, \ldots, r$. Let $\beta_j : (D^n \times I) \to R^{n+2}$ $(j = 1 \ldots, r)$ be embeddings with the properties:

1. $\beta_j(D^n \times I) \cap (S_0^n \cup S_1^n \cup \ldots \cup S_r^n) = \beta_j(D^n \times \partial I)$,
2. $\beta_j(D^n \times I) \cap \beta_{j'}(D^n \times I) = \emptyset$ if $j \neq j'$, and
3. $(\bigcup_{i=0}^r S_i^n) \cup (\bigcup_{j=1}^r \beta_j(D^n \times I))$ is connected.

Then $(\bigcup_{i=0}^r S_i^n - \bigcup_{j=1}^r \beta_j(D^n \times \partial I)) \cup (\bigcup_{j=1}^r \beta_j(\partial D^n \times I))$ is an n-knot in R^{n+2} and denoted by $\mathcal{F}(S_0^n, S_1^n, \ldots, S_r^n \mid \beta_1, \ldots, \beta_r)$. An n-knot K^n is called a *ribbon*

n-knot if K^n and $\mathcal{F}(S_0^n, S_1^n, \ldots, S_r^n \,|\, \beta_1, \ldots, \beta_r)$ are of the same knot type for some $S_0^n, S_1^n, \ldots, S_r^n$ and $\beta_1, \ldots, \beta_r$.

REMARK. We can deform each band $\beta_j(D^n \times I)$ in Definition 1.1 by an ambient isotopy of R^{n+2} so that

$$\beta_j(D^n \times I) \cap S_i^n = \begin{cases} \beta_j(D^n \times \{0\}) & \text{if } i = j-1, \\ \beta_j(D^n \times \{1\}) & \text{if } i = j, \text{ and} \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus throughout this paper we assume that each band of a ribbon n-knot satisfies this condition. We refer to the reader [1].

DEFINITION 1.2. (n-dimensional θ_s-curve) We call an n-complex Θ_s^n an *n-dimensional θ_s-curve* if Θ_s^n is the union of n-disks $B_1^n, \ldots, B_s^n$ with common boundaries and mutually disjoint interiors. If Θ_s^n is embedded in R^{n+2} then we call the $s(s-1)/2$ n-spheres $B_i^n \cup B_j^n$ $(1 \le i < j \le s)$ the *constituent knots* of Θ_s^n.

§2. Main Theorem

The following is the main theorem in this paper.

THEOREM 2.1. *For any collection of $s(s-1)/2$ ribbon n-knots $K_{ij}^n (1 \le i < j \le s)$ there exists an n-dimensional θ_s-curve $\Theta_s^n (= B_1^n \cup \ldots \cup B_s^n)$ in R^{n+2} such that $B_i^n \cup B_j^n$ and K_{ij}^n are of the same knot type for $1 \le i < j \le s$.*

In order to prove Theorem 2.1, we show the following lemma.

LEMMA 2.2. *For any ribbon n-knot K^n in R^{n+2}, there exists an n-dimensional θ_s-curve $\Theta_s^n (= B_1^n \cup \cdots \cup B_s^n)$ in R^{n+2} such that $B_1^n \cup B_2^n$ and K^n are of the same knot type and the other constituent knots of Θ_s^n are unknotted in R^{n+2}.*

PROOF. Let $K^n = \mathcal{F}(S_0^n, S_1^n, \ldots, S_r^n \,|\, \beta_1, \ldots, \beta_r)$. By Remark each $\beta_j(D^n \times I)$ attaches S_{j-1}^n and S_j^n for $j = 1, \ldots, r$, and there exist embeddings $\alpha_i : D^{n+1} \to R^{n+2}$ $(i = 0, 1, \ldots, r)$ such that $\alpha_i(\partial D^{n+1}) = S_i^n$ and $\alpha_i(D^{n+1}) \cap \alpha_{i'}(D^{n+1}) = \emptyset$ for $i \ne i'$. Without loss of generality, we may assume that every intersection $\alpha_i(D^{n+1}) \cap \beta_j(D^n \times I)$ is the union of n-disks. We may further suppose

1. $\alpha_i(\mathrm{Int}\, D^{n+1}) \cap \beta_j(D^n \times I) \subset \alpha_i(\mathrm{Int}\, D_{1/2}^{n+1})$ if $i < j$, and
2. $\alpha_i(\mathrm{Int}\, D^{n+1}) \cap \beta_j(D^n \times I) \subset \alpha_i(D^{n+1} - D_{1/2}^{n+1})$ if $i \ge j$, see Figure 1.

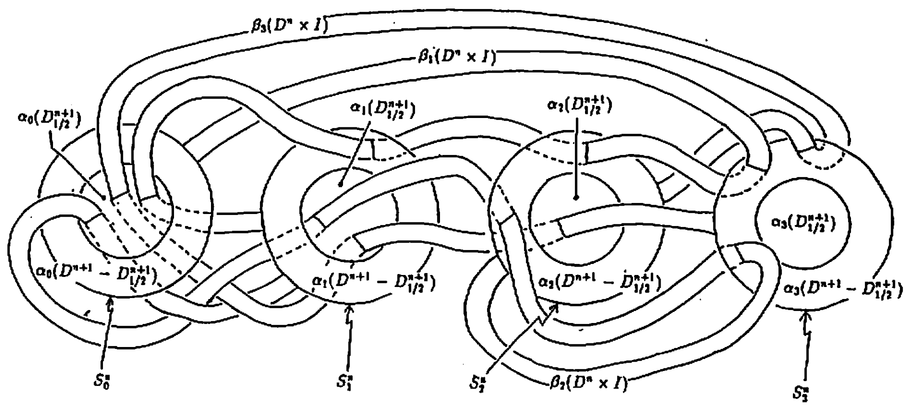

Fig. 1

For each S^n_i $(i = 0,1,\ldots,r)$, let $h_i : S^n_i \times I \to \alpha_i(D^{n+1} - \operatorname{Int} D^{n+1}_{1/2})$ be a homeomorphism such that $h_i(S^n_i \times \{1\}) = S^n_i$. For each $j(= 1,\ldots,r)$, let $\beta'_j : D^n \times I \to R^{n+1}$ be an embedding with its image $\beta'_j(D^n \times I) = h_{j-1}(\beta_j(D^n_{1/2} \times \{0\}) \times I) \cup \beta_j(D^n_{1/2} \times I) \cup h_j(\beta_j(D^n_{1/2} \times \{1\}) \times I)$, see Figure 2. We may assume $h_{j-1}(\beta_j(D^n_{1/2} \times \{0\}) \times \operatorname{Int} I) \cap (\bigcup_{l=1}^{r} \beta_l(D^n \times I)) = \emptyset$ and $h_j(\beta_j(D^n_{1/2} \times \{1\}) \times \operatorname{Int} I) \cap (\bigcup_{l=1}^{r} \beta_l(D^n \times I)) = \emptyset$ $(j = 1,\ldots,r)$.

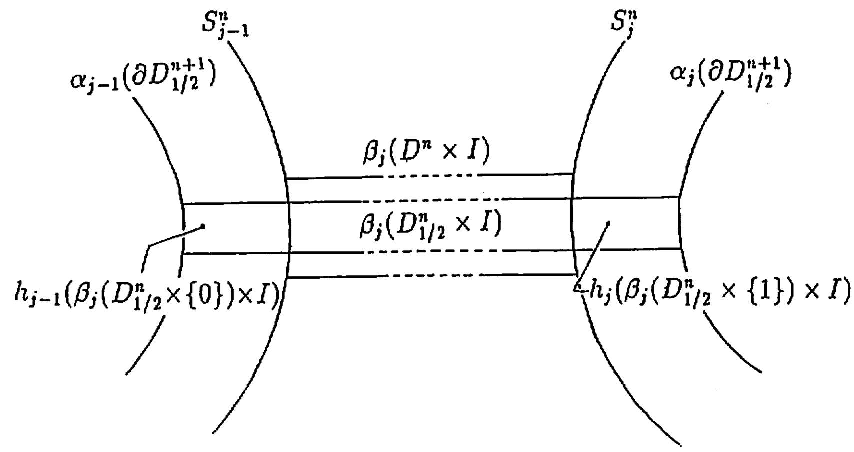

Fig. 2

Let B^n_1 be an n-disk in S^n_0 with $B^n_1 \cap \beta_1(D^n \times \{0\}) = \emptyset$, $B^n_2 = \operatorname{Cl}(K^n - B^n_1)$, and $B^n_3 = (\mathcal{F}(\alpha_0(\partial D^{n+1}_{1/2}),\ \alpha_1(\partial D^{n+1}_{1/2}),\ldots,\alpha_r(\partial D^{n+1}_{1/2}) \,|\, \beta'_1,\ldots,\beta'_r) - h_0(B^n_1 \times \{0\})) \cup h_0(\partial B^n_1 \times I)$, see Figure 3.

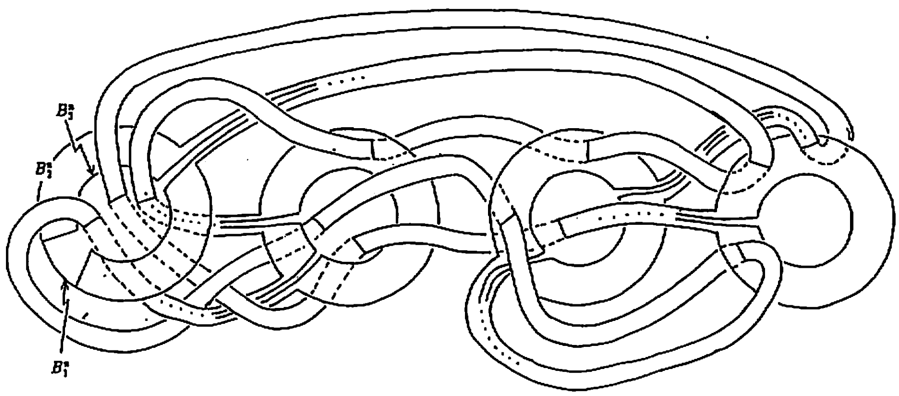

Fig. 3

CLAIM. The n-knots $B_1^n \cup B_3^n$ and $B_2^n \cup B_3^n$ are unknotted in R^{n+2}.

PROOF of CLAIM. By using the Cellular Move Lemma (see 4.15, p. 55 in [3]), we obtain the following:

$$\begin{aligned}
B_1^n \cup B_3^n &\overset{a}{\approx} \mathcal{F}(\alpha_0(\partial D_{1/2}^{n+1}), \alpha_1(\partial D_{1/2}^{n+1}), \dots, \alpha_r(\partial D_{1/2}^{n+1}) \,|\, \beta_1', \dots, \beta_r') \\
&\overset{a}{\approx} \mathcal{F}(\alpha_0(\partial D_{1/2}^{n+1}), \alpha_1(\partial D_{1/2}^{n+1}), \dots, \alpha_{r-1}(\partial D_{1/2}^{n+1}) \,|\, \beta_1', \dots, \beta_{r-1}') \\
&\overset{a}{\approx} \mathcal{F}(\alpha_0(\partial D_{1/2}^{n+1}), \alpha_1(\partial D_{1/2}^{n+1}), \dots, \alpha_{r-2}(\partial D_{1/2}^{n+1}) \,|\, \beta_1', \dots, \beta_{r-2}') \\
&\cdots \\
&\overset{a}{\approx} \mathcal{F}(\alpha_0(\partial D_{1/2}^{n+1}), \alpha_1(\partial D_{1/2}^{n+1}) \,|\, \beta_1') \\
&\overset{a}{\approx} \alpha_0(\partial D_{1/2}^{n+1}),
\end{aligned}$$

where "$\overset{a}{\approx}$" means an ambient isotopy of R^{n+2}. Since $\alpha_0(\partial D_{1/2}^{n+1})$ is unknotted, $B_1^n \cup B_3^n$ is unknotted.

By using the Cellular Move Lemma, we obtain the following:

$$\begin{aligned}
B_2 \cup B_3 = (&\mathcal{F}(S_0^n, S_1^n, \dots, S_r^n \,|\, \beta_1, \dots, \beta_r) \\
&\cup \mathcal{F}(\alpha_0(\partial D_{1/2}^{n+1}), \alpha_1(\partial D_{1/2}^{n+1}) \dots, \alpha_r(\partial D_{1/2}^{n+1}) \,|\, \beta_1', \dots, \beta_r') \\
&- h_0(B_1^n \times \partial I)) \cup h_0(\partial B_1^n \times I)
\end{aligned}$$

$$\overset{a}{\approx} (\mathcal{F}(S_1^n, \dots, S_r^n \,|\, \beta_2, \dots, \beta_r)$$
$$\cup\, \mathcal{F}(\alpha_1(\partial D_{1/2}^{n+1}), \dots, \alpha_r(\partial D_{1/2}^{n+1}) \,|\, \beta_2', \dots, \beta_r')$$
$$- h_1(\beta_1(D_{1/2}^n \times \{1\}) \times \partial I)) \cup h_1(\beta_1(\partial D_{1/2}^n \times \{1\}) \times I)$$
$$\overset{a}{\approx} (\mathcal{F}(S_2^n, \dots, S_r^n \,|\, \beta_3, \dots, \beta_r)$$
$$\cup\, \mathcal{F}(\alpha_2(\partial D_{1/2}^{n+1}), \dots, \alpha_r(\partial D_{1/2}^{n+1}) \,|\, \beta_3', \dots, \beta_r')$$
$$- h_2(\beta_2(D_{1/2}^n \times \{1\}) \times \partial I)) \cup h_2(\beta_2(\partial D_{1/2}^n \times \{1\}) \times I)$$
$$\dots$$
$$\overset{a}{\approx} (S_r^n \cup \alpha_r(\partial D_{1/2}^{n+1}) - h_r(\beta_r(D_{1/2}^n \times \{1\}) \times \partial I))$$
$$\cup\, h_r(\beta_r(\partial D_{1/2}^n \times \{1\}) \times I)$$
$$\overset{a}{\approx} S_r^n.$$

Since S_r^n is unknotted, $B_2^n \cup B_3^n$ is unknotted. □

Thus $B_1^n \cup B_2^n \cup B_3^n$ is an n-dimensional θ_3-curve such that $B_1^n \cup B_3^n$ and $B_2^n \cup B_3^n$ are unknotted in R^{n+2} and $B_1^n \cup B_2^n = K^n$. Taking n-disks $B_4, B_5, \dots, B_s$ such that

1. $K^n \cap B_i^n = \partial B_3^n = \partial B_i^n$ $(i = 4, 5, \dots, s)$, and
2. $B_i^n \cup B_j^n$ bounds an $(n+1)$-disk whose interior does not intersect $K^n (3 \leq i < j \leq s)$,

we obtain Lemma 2.2. □

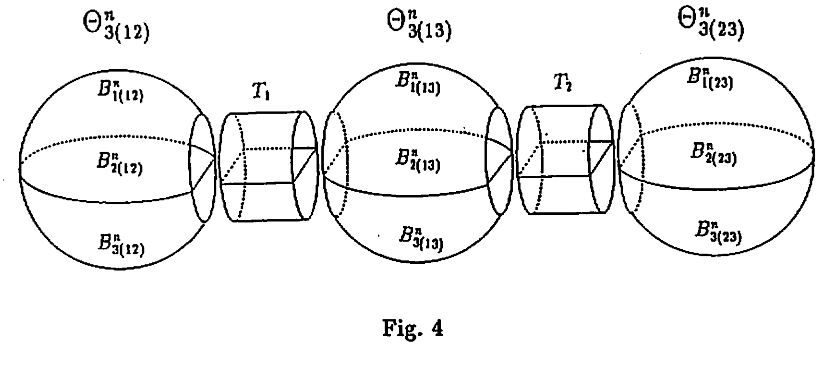

Fig. 4

Proof of Theorem 2.1. We consider $s(s-1)/2$ ribbon n-knots $K_{ij}^n (1 \leq i < j \leq s)$. By Lemma 2.2, for each K_{ij}^n, there exists an n-dimensional θ_s-curve $\Theta_{s(ij)}^n (= B_{1(ij)}^n \cup \dots \cup B_{s(ij)}^n)$ such that $B_{i(ij)}^n \cup B_{j(ij)}^n$ and K_{ij}^n are of the same knot type, and the other constituent knots are unknotted in R^{n+2}. Let $T_k(k = 1, \dots, s(s-1)/2 - 1)$ be homeomorphic to $\Theta_s^{n-1} \times I$. Connecting the n-dimensional θ_s-curves $\Theta^n_{s(ij)}(1 \leq i < j \leq s)$ by $T_k(k = 1, \dots, s(s-1)/2 - 1)$ suitably, see Figure 4, we obtain Theorem 2.1. □

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