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Symmetry of Links and Classification of Lens Spaces

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Abstract. We give a concise proof of a classification of lens spaces up to orientation-preserving homeomorphisms. The chief ingredient in our proof is a study of the Alexander polynomial of 'symmetric' links in S^3.

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Let T_1 and T_2 be solid tori, and let m_i and l_i be the meridian and longitude of T_i $(i = 1, 2)$ oriented in such a way that the pair (l_i, m_i) yields the positive orientation of ∂T_i. The *lens space* $L(p, q)$ is a 3-manifold that is obtained from T_1 and T_2 by identifying their boundaries in such a way that $-m_2 = pl_1 + qm_1$ and $-l_2 = \bar{q}l_1 + rm_1$, where $(p, q) = 1$ and $q\bar{q} - pr = 1$.

In 1935, Reidemeister [12] classified lens spaces up to orientation-preserving PL homeomorphisms. This classification was generalized to the topological category with the proof of the Hauptvermutung by Moise in 1952 [9]. Meanwhile, Fox had outlined an approach to classification up to homeomorphisms which would not require the Hauptvermutung; see [5, Problem 2], [6]. This was implemented later by Brody [3]. Turaev showed in his PhD thesis [14] that Brody method/invariant can be understood in the language of Reidemeister torsion (compare also [7, 13]). We refer the reader to [8] for history of classifications of lens spaces.

In this paper, we give a concise proof of a classification of lens spaces up to orientation-preserving homeomorphisms. Our method is motivated by that of Fox–Brody. While the chief ingredient in their proof was a study of the Alexander polynomial of knots in lens spaces, we study the Alexander polynomial of 'symmetric' links in S^3.

For an oriented 3-manifold M with finite first homology group, the *linking form*

$$\mathrm{lk}_M\colon H_1(M; \mathbb{Z}) \times H_1(M; \mathbb{Z}) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

is defined as follows [1, 2]. Let x and y be 1-cycles in M that represent elements $[x]$ and $[y]$ of $H_1(M; \mathbb{Z})$ respectively. Suppose that nx bounds a 2-chain c for some $n \in \mathbb{Z}$. Then

$$\mathrm{lk}_M([x],[y]) = \frac{c \cdot y}{n} \in \mathbb{Q}/\mathbb{Z},$$

where $c \cdot y$ is the intersection number of c and y.

Let $\Delta_K(t)$ be the *Conway-normalized Alexander polynomial* of K, i.e., $\Delta_K(t) = \nabla_K(t^{-1/2} - t^{1/2})$, where $\nabla_K(z)$ is the Conway polynomial.

THEOREM 1. *Let $\rho: S^3 \longrightarrow L(p,q)$ be the p-fold cyclic cover and K a knot in $L(p,q)$ that represents a generator of $H_1(L(p,q);\mathbb{Z})$. If $\Delta_{\rho^{-1}(K)}(t) = 1$, then* $\mathrm{lk}_{L(p,q)}([K],[K]) = -q/p$ *or* $= -\bar{q}/p$ *in* $\mathbb{Q}/\mathbb{Z}$.

Before proving Theorem 1, we obtain the classification of lens spaces as its corollary.

COROLLARY 2. *Two lens spaces $L(p,q)$ and $L(p,q')$ are equivalent up to orientation-preserving homeomorphisms if and only if $q \equiv q'$* (mod p) *or $qq' \equiv 1$* (mod p).

Proof. The 'if' part of the corollary is well known and easy to see. It is enough to show the 'only if' part. Since $pl_1 + qm_1$ bounds a meridian disk D_2 of T_2, pl_1 bounds a 2-chain $-(qD_1 \cup D_2)$, where qD_1 is a disjoint union of q copies of a meridian disk D_1 of T_1. So we have $\mathrm{lk}_{L(p,q)}([l_1],[l_1]) = -(qD_1 \cup D_2) \cdot l_1/p = -q/p$. Note that $\rho^{-1}(l_1)$ is the trivial knot, and hence $\Delta_{\rho^{-1}(l_1)}(t) = 1$. Suppose that there is an orientation-preserving homeomorphism $f: L(p,q) \longrightarrow L(p,q')$. Then $\mathrm{lk}_{L(p,q')}([f(l_1)],[f(l_1)]) = \mathrm{lk}_{L(p,q)}([l_1],[l_1])$ and $\Delta_{\rho^{-1}(f(l_1))}(t) = \Delta_{\rho^{-1}(l_1)}(t)$. Therefore, by Theorem 1, we have $q/p \equiv q'/p$ or $\equiv \bar{q}'/p$ (mod 1). This implies that $q \equiv q'$ (mod p) or $qq' \equiv 1$ (mod p). □

Remark. Since $\mathrm{lk}_{L(p,q)}([nl_1],[nl_1]) = -n^2 q/p$, the set $\{n^2 q/p | 1 \leqslant n < p\}$ of rational numbers is an invariant of $L(p,q)$ up to orientation-preserving homotopy. A homotopy classification of lens spaces was obtained by Whitehead [15]: $L(p,q)$ and $L(p,q')$ are equivalent up to orientation-preserving homotopy if and only if $qq' \equiv n^2$ (mod p) for some integer n. The necessity of the condition is given by this invariant.

In order to prove Theorem 1, we need some preliminaries. In the following lemma, we consider the Alexander polynomial of links with a certain kind of symmetry.

LEMMA 3. *Let r be a prime and s a positive integer. Let L be an r^s-periodic link in S^3 and L' a link obtained from L by changing a set of crossings that is the $\mathbb{Z}_{r^s}$-orbit of a single crossing in the periodic diagram of L. For an integer q, let $L(q)$* (*resp. $L'(q)$*) *be a link obtained from L* (*resp. L'*) *by adding $-q$-full twists as illustrated in Figure* 1; *equivalently we perform a $1/q$-surgery along the fixed point set of the $\mathbb{Z}_{r^s}$-action. Then*

$$\Delta_{L(q)}(t) \equiv \Delta_{L'(q)}(t) \bmod (t^{r^s} - 1, r).$$

Proof. The full twist is in the center of the algebra of tangles. Thus, in Figure 1(b), a tangle T and the full-twists part are commutative. Therefore by arguments similar to that in the proof of Lemma 2.3 in [10] (compare also [4, 11]), we obtain

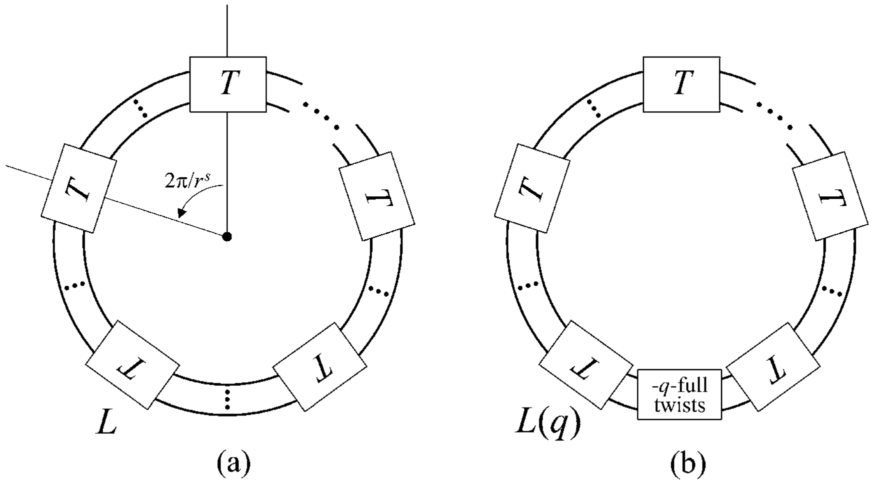

Figure. 1

$$\Delta_{L(q)}(t) \equiv \Delta_{L'(q)}(t) \bmod((t^{-1/2} - t^{1/2})^{r^s}, r).$$

Since

$$(t^{-1/2} - t^{1/2})^{r^s} \equiv t^{-r^s/2} - t^{r^s/2} \pmod r,$$

we have the conclusion. □

A link in S^3, the universal cover of a lens space, that covers a knot in the lens space has the above-mentioned symmetry. By Lemma 3, we obtain the following proposition.

PROPOSITION 4. *Let ρ: $S^3 \longrightarrow L(p, q)$ be the p-fold cyclic cover and let K and K′ be knots in $L(p, q)$. If K and K′ are homologous in $L(p, q)$ and if p is divisible by the sth power r^s of a prime integer r, then*

$$\Delta_{\rho^{-1}(K)}(t) \equiv \Delta_{\rho^{-1}(K')}(t) \bmod(t^{r^s} - 1, r).$$

Proof. We may assume that both K and K' are contained in T_1. We note that the p-fold cover S^3 is obtained from $\tilde{T}_1$ and $\tilde{T}_2$ by identifying their boundaries in such a way that $-\tilde{m}_2 = \tilde{l}_1 + q\tilde{m}_1$, where $\tilde{T}_i$ is a solid torus that is the p-fold cover of T_i, and $\tilde{m}_i$ and $\tilde{l}_i$ are the meridian and longitude of $\tilde{T}_i$ $(i = 1, 2)$. If two knots in $L(p, q)$ are homologous, then they are also homotopic. So they differ in a finite number of crossings. Thus it suffices to consider the case in which K' is obtained from K by changing a single crossing c. Then $\rho^{-1}(K)$ and $\rho^{-1}(K')$ are p-periodic in $\tilde{T}_1$, $\rho^{-1}(K')$ is obtained from $\rho^{-1}(K)$ by changing the crossings $\rho^{-1}(c)$, and a covering translation ϕ of $\tilde{T}_1$ generates the $\mathbb{Z}_p$-action on $\tilde{T}_1$. Set $p = ur^s$. Then we note that

$$\rho^{-1}(c) = \{c_{11}, \ldots, c_{1u}\} \cup \cdots \cup \{c_{r^s1}, \ldots, c_{r^su}\},$$

where $\phi^{j-1}(c_{k1}) = c_{kj}$ and $\phi^{u(k-1)}(c_{1j}) = c_{kj}$ for $k = 1, \dots, r^s$, $j = 1, \dots, u$. This implies that there is a sequence

$$\rho^{-1}(K) = L_0, \dots, L_u = \rho^{-1}(K')$$

of r^s-periodic links in $\tilde{T}_1$ such that L_j is obtained from L_{j-1} by changing the crossings $c_{1j}, \dots, c_{r^s j}$ ($j = 1, \dots, u$). By Lemma 3, we have

$$\Delta_{L_{j-1}}(t) \equiv \Delta_{L_j}(t) \bmod (t^{r^s} - 1, r).$$

This completes the proof. □

Proof of Theorem 1. Let K_n be a knot in $\partial T_1 \subset L(p, q)$ that represents $n[l_1] + [m_1] \in H_1(L(p, q); \mathbb{Z})$. We may assume that K is homologous to K_n for some n ($(n, p) = 1$). Then $\rho^{-1}(K_n)$ is the $(n, p - qn)$-torus knot since S^3 is obtained from $\tilde{T}_1$ and $\tilde{T}_2$ by identifying their boundaries such that $-\tilde{m}_2 = \tilde{l}_1 + q\tilde{m}_1$. It is well known that

$$\Delta_{\rho^{-1}(K_n)} = t^{-g} \frac{(1 - t)(1 - t^{n(p-qn)})}{(1 - t^n)(1 - t^{p-qn})},$$

where $g = (n(p - qn) + 1 - (p - qn + n))/2$. By Proposition 4,

$$t^{-g} \frac{(1 - t)(1 - t^{n(p-qn)})}{(1 - t^n)(1 - t^{p-qn})} \equiv 1 \bmod (t^{r^s} - 1, r),$$

where r^s is a divisor of p and r is a prime integer. Set $a = p - qn$. (Note that $a \equiv -qn$ (mod r^s) and $(p, a) = 1$.) Then we have

$$1 + t^{na+1} + t^{a+g} + t^{n+g} \equiv t^g + t^{a+n+g} + t^{na} + t \bmod (t^{r^s} - 1, r).$$

By elementary calculations, we obtain $n^2 \equiv 1$ or $\equiv \bar{q}^2$ (mod r^s). So we have $n^2 \equiv 1$ or $\equiv \bar{q}^2$ (mod p). Since $\mathrm{lk}_{L(p,q)}([K], [K]) = -n^2 q/p$, we have the required result. □

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