

Realization of knots and links in a spatial graph

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Abstract

For a graph G , let Γ be either the set Γ_1 of cycles of G or the set Γ_2 of pairs of disjoint cycles of G . Suppose that for each $\gamma \in \Gamma$, an embedding $\phi_\gamma : \gamma \rightarrow S^3$ is given. A set $\{\phi_\gamma \mid \gamma \in \Gamma\}$ is realizable if there is an embedding $f : G \rightarrow S^3$ such that the restriction map $f|_\gamma$ is ambient isotopic to ϕ_γ for any $\gamma \in \Gamma$. A graph is adaptable if any set $\{\phi_\gamma \mid \gamma \in \Gamma_1\}$ is realizable. In this paper, we have the following three results:

(1) For the complete graph K_5 on 5 vertices and the complete bipartite graph $K_{3,3}$ on $3 + 3$ vertices, we give a necessary and sufficient condition for $\{\phi_\gamma \mid \gamma \in \Gamma_1\}$ to be realizable in terms of the second coefficient of the Conway polynomial.

(2) For a graph in the Petersen family, we give a necessary and sufficient condition for $\{\phi_\gamma \mid \gamma \in \Gamma_2\}$ to be realizable in terms of the linking number.

(3) The set of non-adaptable graphs all of whose proper minors are adaptable contains eight specified planar graphs. © 2001 Elsevier Science B.V. All rights reserved.

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1. Definitions and results

In [1] it is shown that any spatial embedding of the complete graph on six (respectively seven) vertices contains a pair of nontrivially linked cycles (respectively a nontrivially knotted cycle). We refer the reader to [13,6,12,10,19,11,20,21] for related works. On the contrary it is shown in [4] that any $n(n - 1)/2$ knot types are realized by a spatial embedding of θ_n at once where θ_n is the graph on two vertices and n edges joining

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them. We refer the reader to [23,5,24,26,19,20] for related works. These results show both dependence and independence of the knot types in a spatial graph. Thus we are interested in whether or not a given collection of knot types is realized by a spatial graph at once. In this paper we give a complete characterization of the knots contained in a spatial embedding of the complete graph K_5 on five vertices, and that of the complete bipartite graph $K_{3,3}$ on $3 + 3$ vertices (Theorem 1.1). Then we give a complete characterization of the 2-component links contained in a spatial embedding of a graph that is one of seven specified graphs including the complete graph K_6 on six vertices, the complete tripartite graph $K_{3,3,1}$ on $3 + 3 + 1$ vertices and the Petersen graph (Theorem 1.2). If a graph G contains n cycles and any collection of n knots are realized by the spatial embedding of G at once then G is called *adaptable*. We find several minimal non-adaptable graphs (Theorems 1.3, 1.4).

Throughout this paper, we work in the piecewise linear category. Let G be a finite graph. We consider G as a topological space in the usual way.

Let $\text{SE}(G)$ be the set of all embeddings of G into the 3-sphere S^3 . We call an element of $\text{SE}(G)$ a *spatial embedding* of a graph, or simply a *spatial graph*. Let Γ be a set of subgraphs of G . Suppose that for each $\gamma \in \Gamma$, $\phi_\gamma \in \text{SE}(\gamma)$ is given. We say that a set of embeddings $\{\phi_\gamma \in \text{SE}(\gamma) \mid \gamma \in \Gamma\}$ is *realizable* if there is an element $f \in \text{SE}(G)$ such that the restriction map $f|_\gamma$ is ambient isotopic to ϕ_γ for all $\gamma \in \Gamma$. A *cycle* is a subgraph of G that is homeomorphic to a circle. A *k-cycle* is a cycle that contains exactly k vertices. Let $\Gamma_1(G)$ be the set of all cycles of G and $\Gamma_2(G)$ the set of all pairs of disjoint cycles of G . In this paper all knots and links are oriented and the components of a link have an ordering. We naturally identify an embedding of a cycle into S^3 with a knot and an embedding of a pair of disjoint cycles into S^3 with a 2-component link. We sometimes do not mention the orientations and the orderings explicitly so long as no ambiguity occurs. Let $a_2(J)$ be the second coefficient of the Conway polynomial of a knot J and $\text{lk}_2(L)$ the modulo 2 reduction of the linking number $\text{lk}(L)$ of a 2-component link L . Note that they are independent of the choice of orientation and ordering.

Theorem 1.1. *Let G be the complete graph K_5 on 5 vertices (Fig. 1(1)) or the complete bipartite graph $K_{3,3}$ on $3 + 3$ vertices (Fig. 1(2)). Let C_1 be the set of 4-cycles of G and C_2 the set of 5-cycles (respectively 6-cycles) of K_5 (respectively $K_{3,3}$). Then a set of embeddings $\{\phi_\gamma \in \text{SE}(\gamma) \mid \gamma \in \Gamma_1(G)\}$ is realizable if and only if there is an integer m such that*

$$\sum_{\gamma \in C_2} a_2(\phi_\gamma(\gamma)) - \sum_{\gamma \in C_1} a_2(\phi_\gamma(\gamma)) = \frac{m(m-1)}{2}.$$

The ‘only if’ part of Theorem 1.1 was shown by Motohashi and the first author [8, Theorems 3.2 and 3.3].

Theorem 1.2. *Let G be one of the seven graphs in Figs. 2(1), (2), ..., (7). Then a set of embeddings $\{\phi_\gamma \in \text{SE}(\gamma) \mid \gamma \in \Gamma_2(G)\}$ is realizable if and only if*

$$\sum_{\gamma \in \Gamma_2(G)} \text{lk}_2(\phi_\gamma(\gamma)) \equiv 1 \pmod{2}.$$

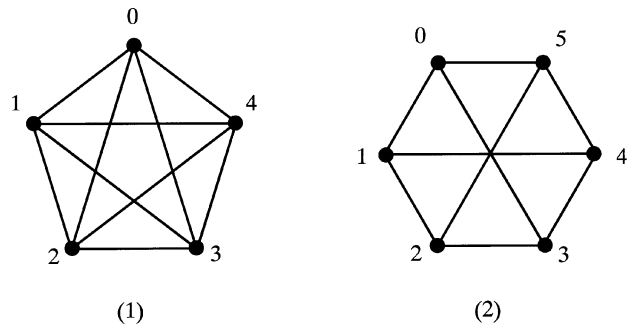


Fig. 1.

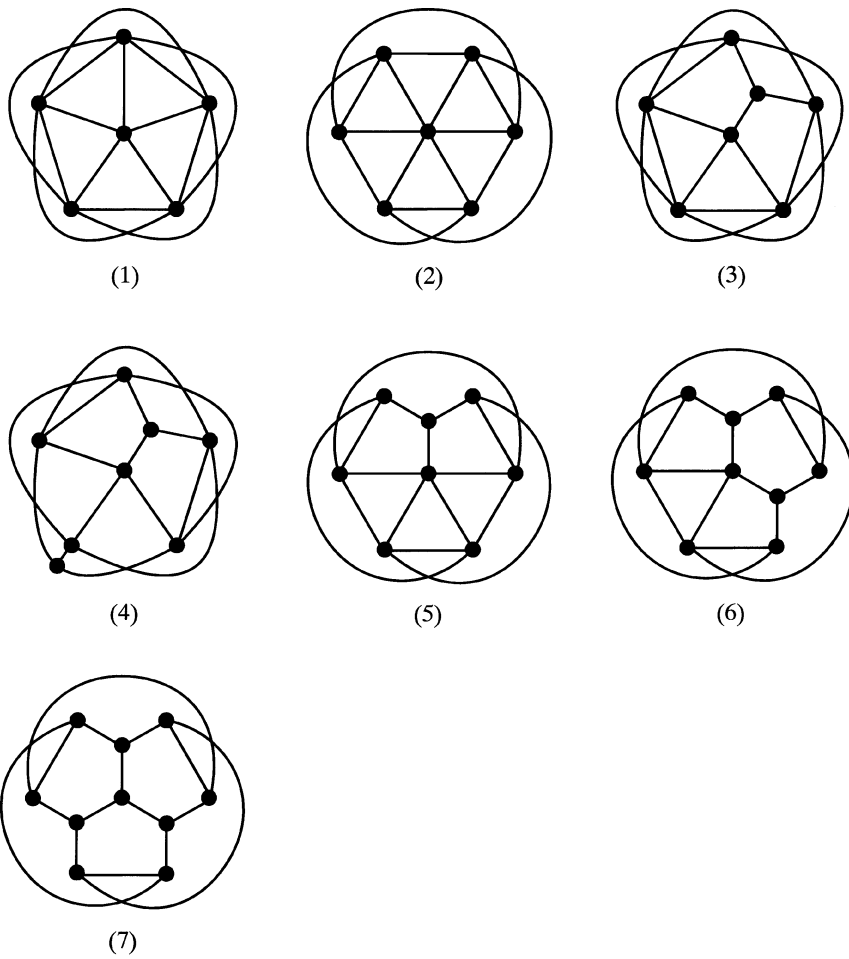


Fig. 2.

The seven graphs in Figs. 2(1), (2), \dots , (7) are all ones obtainable from K_6 by a sequence of interchanges of triangles with vertices of degree 3 (called $Y\Delta$ -exchanges and ΔY -exchanges), and the set of the seven graphs is called the *Petersen family* [12].

The ‘only if’ part in the case $G = K_6$ is shown in [1, Proof of Theorem 1], and those in the other cases are shown by the arguments similar to that in [1, Proof of Theorem 1].

A graph G is called *adaptable* if any set of embeddings $\{\phi_\gamma \in \text{SE}(\gamma) \mid \gamma \in \Gamma_1(G)\}$ is realizable.

Let G be a graph and $e = uv$ an edge of G that is not a loop. The *edge contraction* of G along e is the graph obtained from $G - \text{int } e$ by identifying the ends u and v , and is denoted by G/e . A graph H is a *minor* of a graph G , denoted by $H \leq G$, if there is a subgraph G' of G and its edges e_1, e_2, \dots, e_n such that H is obtained from G' by a sequence of edge contractions along e_1, e_2, \dots, e_n , i.e.,

$$H = (\dots((G'/e_1)/e_2)/\dots)/e_n.$$

Any subgraph of G is a minor of G . If $H \leq G$ and $H \neq G$, then we call H a *proper minor* of G .

Note that if $H \leq G$ then there is a natural injection from $\Gamma_1(H)$ to $\Gamma_1(G)$. Therefore if $H \leq G$ and G is adaptable, then H is adaptable. We define the *obstruction set* Ω for adaptability to be the set of non-adaptable graphs all of whose proper minors are adaptable. Let Ω_0 be the set of planar graphs in Ω and $\Omega_1 = \Omega \setminus \Omega_0$. It is known that the following theorem holds.

Theorem 1.3 (Yasuhara [26]). $\Omega_1 = \{K_5, K_{3,3}\}$.

The fact that K_5 and $K_{3,3}$ are not adaptable was shown by Motohashi and the first author [8]. In [26], the second author showed that all proper minors of $K_5, K_{3,3}$ are adaptable. Therefore the theorem above follows from Kuratowski’s Theorem [7].

Theorem 1.4. *Let G_1, G_2, \dots, G_8 be graphs as in Figs. 3(1), (2), \dots , (8), respectively. Then $\{G_1, G_2, G_3, \dots, G_8\} \subset \Omega_0$.*

The six graphs in Figs. 3(1), (2), \dots , (6) are all ones obtainable from 3-cube (Fig. 3(1)) by $Y\Delta$ -exchanges.

The following theorem is a key to proving that some proper minors of graphs in Theorem 1.4 are adaptable.

Theorem 1.5. *A graph $H(l, m, n)$ as in Fig. 4 is adaptable.*

In [4] Kinoshita showed that the θ_n -curves ($n \geq 3$) are adaptable, and in [23] Yamamoto showed that the complete graph K_4 on four vertices is adaptable. Since θ_n -curves and K_4 are minors of some $H(l, m, n)$, these results also follow from Theorem 1.5. In [26], the second author gave a sufficient condition for graphs to be adaptable. Unfortunately, the graph $H(l, m, n)$ in Theorem 1.5 does not satisfy this condition. Thus we need a new technique to prove Theorem 1.5.

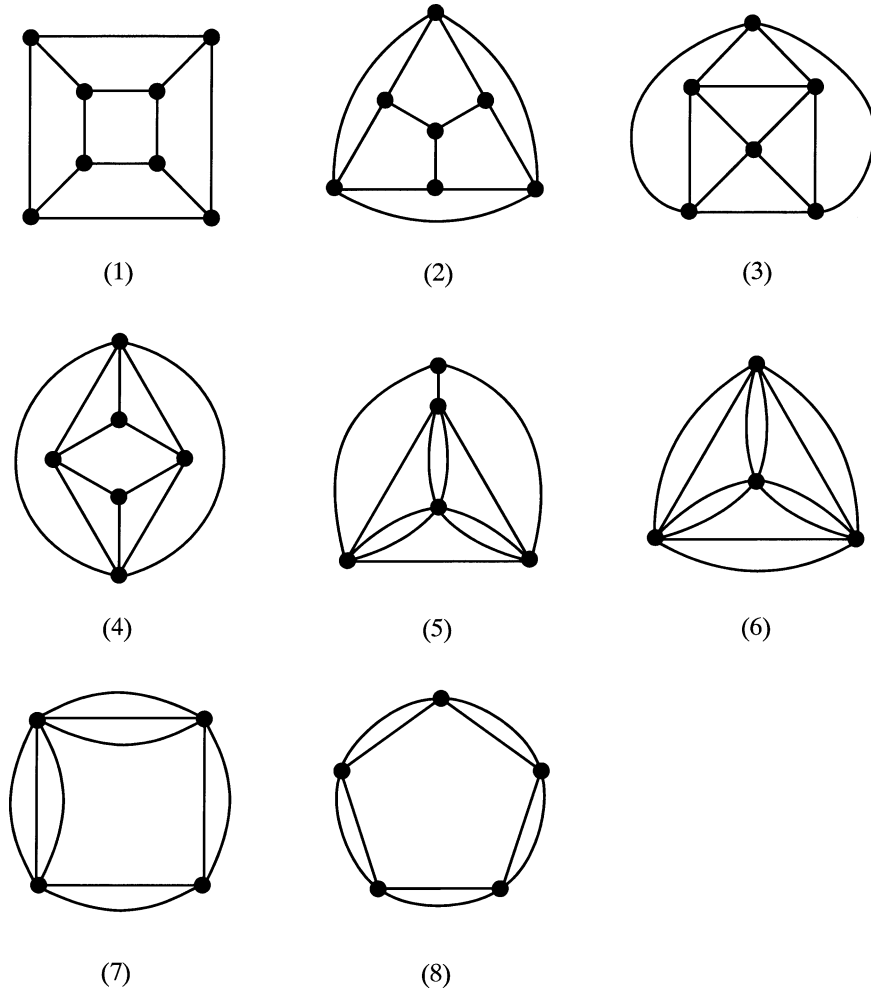


Fig. 3.

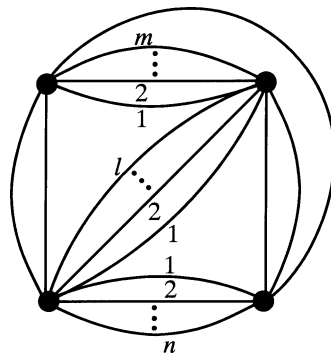


Fig. 4.

2. Proof of Theorem 1.1

Let B^3 be the unit 3-ball. We assume that B^3 has a fixed orientation. A *tangle* is a disjoint union of properly embedded arcs in B^3 . A tangle is *trivial* if it is contained in a properly embedded disk in B^3 . All tangles in this paper are assumed to be trivial. Let T and S be tangles, and let t_1, t_2, \dots, t_k and s_1, s_2, \dots, s_k be the components of T and S , respectively. Suppose that $\partial t_i = \partial s_i$ ($i = 1, 2, \dots, k$). Then we call the ordered pair (T, S) a *local move*. Two local moves (T, S) and (T', S') are *equivalent* if there exists an orientation preserving homeomorphism $h: B^3 \rightarrow B^3$ such that $h(T)$ and T' are ambient isotopic relative to ∂B^3 and $h(S)$ and S' are ambient isotopic relative to ∂B^3 . We consider local moves up to this equivalence. A local move (T, S) is *trivial*, if (T, S) is equivalent to a local move (T, T) . Let (T, S) be a local move, and let t_1, t_2, \dots, t_k and s_1, s_2, \dots, s_k be the components with $\partial t_i = \partial s_i$ ($i = 1, 2, \dots, k$) of T and S , respectively. We call (T, S) a *k-component Brunnian local move* if each local move $(T - t_i, S - s_i)$ is trivial ($i = 1, 2, \dots, k$).

Examples.

- (1) The local move as in Fig. 5(1) is a 2-component Brunnian local move. This move is called a *crossing change*.
- (2) The local move as in Fig. 5(2) is a 3-component Brunnian local move. This move is defined by Murakami and Nakanishi [9] and is called the *delta move*.
- (3) The local move as in Fig. 5(3) is a 4-component Brunnian local move. This move is defined by Habiro [2,3] and is called the *clasp-pass move*.

A local move (S, T) is called the *inverse* of a local move (T, S) . It is not hard to see that the inverses of the local moves in examples are equivalent to themselves.

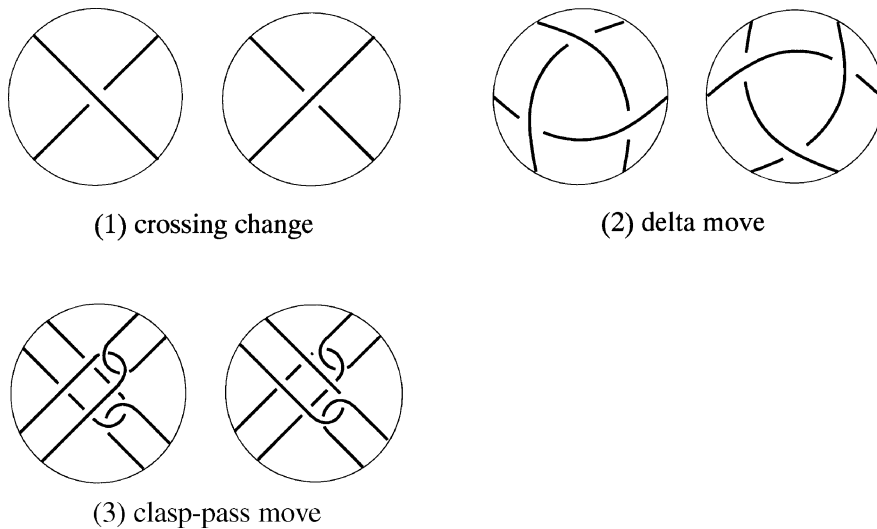


Fig. 5.

Let G be a finite graph. We denote the set of the vertices of G by $V(G)$. Let $f, g : G \rightarrow S^3$ be embeddings. We say that g is obtained from f by a local move (S, T) if there is an orientation preserving embedding $\varphi : B^3 \rightarrow S^3$ such that

- (i) if $f(x) \neq g(x)$ then both $f(x)$ and $g(x)$ are contained in $\varphi(\text{int } B^3)$,
- (ii) $f(V(G)) = g(V(G))$ is disjoint from $\varphi(B^3)$, and
- (iii) $f(G) \cap \varphi(B^3) = \varphi(S)$ and $g(G) \cap \varphi(B^3) = \varphi(T)$.

Theorem 2.1 (Murakami and Nakanishi [9]). *Let J_1 and J_2 be knots. Then J_1 is obtained from J_2 by delta moves and ambient isotopies.*

Theorem 2.2 (Habiro [2,3]). *Let J_1 and J_2 be knots. Then J_1 is obtained from J_2 by clasp-pass moves and ambient isotopies if and only if $a_2(J_1) = a_2(J_2)$.*

See also [22] for a proof of Theorem 2.2.

Lemma 2.3. *Let G be a graph and γ_0 a cycle of G . Let e_1, e_2, \dots, e_n be edges in γ_0 (possibly $e_i = e_j$, $i \neq j$). For a spatial embedding $f \in \text{SE}(G)$ and a knot J if J is obtained from $f|_{\gamma_0}$ by an n -component Brunnian local move, then there is a spatial embedding $g \in \text{SE}(G)$ such that*

- (i) $g|_{\gamma_0}$ is ambient isotopic to J ,
- (ii) $g|_{\gamma}$ is ambient isotopic to $f|_{\gamma}$ if a cycle γ does not contain $\bigcup e_i$, and
- (iii) g is obtained from f by an n -component Brunnian local move and ambient isotopies.

Proof. For an n -component Brunnian local move $(t_1 \cup t_2 \cup \dots \cup t_n, s_1 \cup s_2 \cup \dots \cup s_n)$ with an orientation preserving embedding $\varphi : B^3 \rightarrow S^3$ performed on $f|_{\gamma_0}$, we may assume, after renumbering the suffices if necessary, that $\varphi(t_i) \subset f(\text{int } e_i)$ ($i = 1, 2, \dots, n$). Since the tangle $(B^3, t_1 \cup t_2 \cup \dots \cup t_n)$ is trivial, we can deform f by an ambient isotopy so that $f(G) \cap \varphi(B^3) = \varphi(t_1 \cup t_2 \cup \dots \cup t_n)$. This implies that an n -component Brunnian local move performed on $f|_{\gamma_0}$ can be extended to f . Thus we have a new spatial embedding g obtained from f by an n -component Brunnian local move and ambient isotopies such that $g|_{\gamma_0}$ is ambient isotopic to J . By the definition of n -component Brunnian local move, we have that $g|_{\gamma}$ is ambient isotopic to $f|_{\gamma}$ if a cycle γ does not contain $\bigcup e_i$. \square

Since the delta move is a 3-component Brunnian local move, by Theorem 2.1 and Lemma 2.3, we have the following lemma.

Lemma 2.4. *Let G be a graph and C^1, C^2, \dots, C^n sets of cycles of G with $C^i \cap C^j = \emptyset$ ($i \neq j$). Set $C^1 \cup C^2 \cup \dots \cup C^n = \{\gamma_1, \gamma_2, \dots, \gamma_m\}$. If for any k ($= 1, 2, \dots, n$) and for any $\gamma \in C^k$, there are three edges e_1, e_2 and e_3 (possibly $e_i = e_j$, $i \neq j$) of G such that γ is a unique cycle in $C^1 \cup C^2 \cup \dots \cup C^k$ that contains $e_1 \cup e_2 \cup e_3$, then for any spatial embedding $f \in \text{SE}(G)$ and for any knots J_1, J_2, \dots, J_m , there is a spatial embedding $g \in \text{SE}(G)$ such that*

- (i) $g|\gamma_i$ is ambient isotopic to J_i ($i = 1, 2, \dots, m$) and
- (ii) g is obtained from f by delta moves and ambient isotopies.

In particular, if $C^1 \cup C^2 \cup \dots \cup C^n$ is equal to the set $\Gamma_1(G)$ of all cycles of G , then G is adaptable.

Remarks.

- (1) Motohashi and the first author [8, Theorem 4.1] proved that a planar graph that does not contain disjoint cycles nor vertices of degree less than two is a *generalized bouquet*, or is homeomorphic to a subgraph of either a *multi-spoke wheel* or a *double trident*. See [8] for the definitions of ‘generalized bouquet’, ‘multi-spoke wheel’ and ‘double trident’. Since, by Lemma 2.4, we find that generalized bouquets, multi-spoke wheels and double tridents are adaptable, we have that planar graphs that does not contain disjoint cycles are adaptable.
- (2) Since for any cycle γ of K_5 , there are three or four edges such that γ is a unique cycle of K_5 that contains these edges, by Theorem 1.1 and Lemma 2.4, an n -component Brunnian local move with $n \geq 4$ is not an unknotting operation. That is, for any $n \geq 4$ there exists a knot that cannot be obtained from a trivial knot by n -component Brunnian local moves and ambient isotopies. In fact we have that any n -component Brunnian local move with $n \geq 4$ does not change order 2 Vassiliev invariants of knots, see [15].

Proof of Theorem 1.1. The ‘only if’ part of Theorem 1.1 is shown by Motohashi and the first author [8, Theorems 3.2 and 3.3]. We show the ‘if’ part of this theorem. In the case that $G = K_5$, we assign the labels $0, 1, 2, 3, 4 \in \mathbb{Z}/5\mathbb{Z}$ to the vertices of K_5 as in Fig. 1(1). We classify the cycles of K_5 into the following eight classes; the set C^0 of 3-cycles, the set C^k of cycles as in Fig. 6(k) ($k = 1, 2, \dots, 7$). For a spatial embedding $f \in \text{SE}(K_5)$, we set

$$\alpha(f) = \sum_{\gamma \in C_2} a_2(f(\gamma)) - \sum_{\gamma \in C_1} a_2(f(\gamma)).$$

For an integer m , let $f_m \in \text{SE}(K_5)$ be a spatial embedding as in Fig. 7. It is not hard to see that $\alpha(f_m) = m(m-1)/2$, see [17, Example 3.2].

Claim. For two spatial embeddings $f, g \in \text{SE}(K_5)$, if f is obtained from g by delta moves, then $\alpha(f) = \alpha(g)$.

Proof. Two spatial embeddings are *vertex homotopic* [8] (or *weak link homotopic* [16]) if one is obtained from the other by crossing changes of *adjacent* edges and ambient isotopies, where adjacent edges are edges that have a common end vertex. In [17, Example 3.2], the first author showed that α is a vertex homotopy invariant. By [8, Main Theorem and Theorem 4.2], $f, g \in \text{SE}(K_5)$ are *homologous* if and only if they are vertex homotopic. Hence α is a graph-homology invariant. Here ‘graph-homology’ is an equivalence relation on the set of spatial graphs defined by the first author [16]. The definition will be given in Section 4. Motohashi and the first author [8, Theorem 1.3]

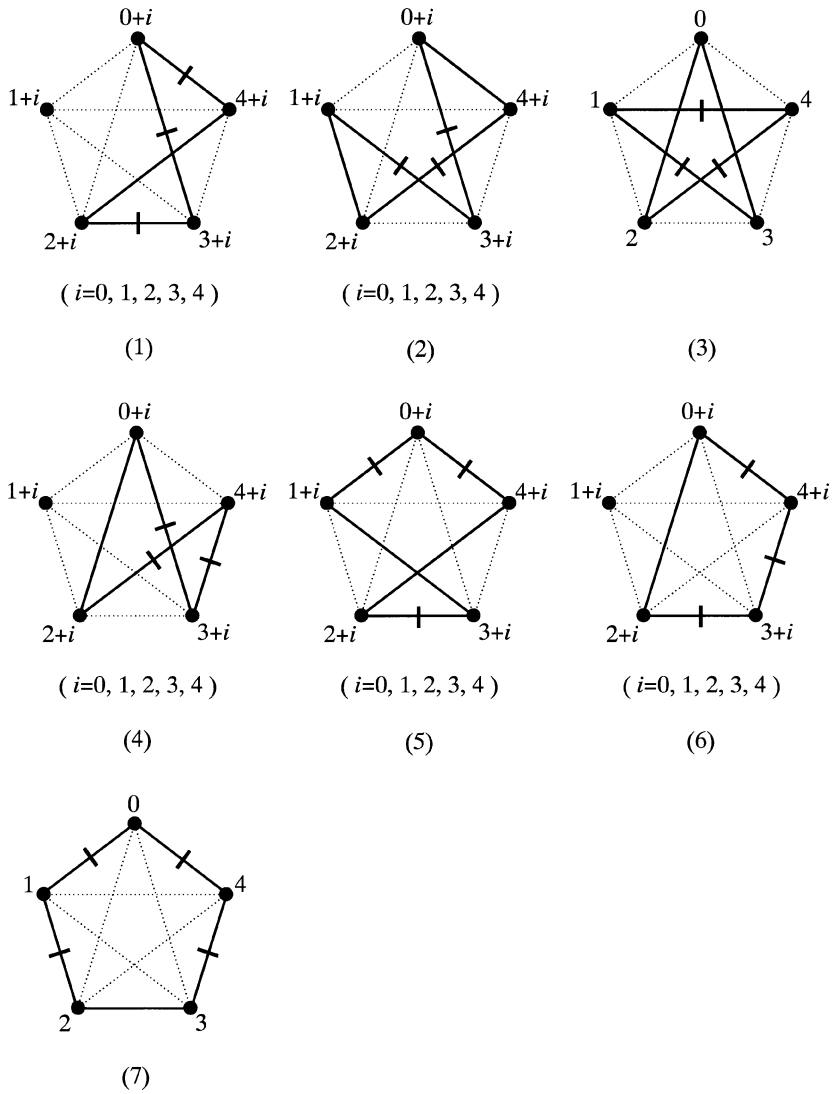


Fig. 6.

showed that two spatial graphs are homologous if and only if one is obtained from the other by delta moves and ambient isotopies, see Theorem 4.1. This completes the proof. \square

Let γ be a cycle in C^k ($k = 0, 1, 2, \dots, 6$). Let $\{e_\gamma^1, e_\gamma^2, e_\gamma^3\}$ be a set of edges of γ checked in Fig. 6(k). Since γ is a unique cycle in $C^0 \cup C^1 \cup \dots \cup C^k$ containing $e_\gamma^1 \cup e_\gamma^2 \cup e_\gamma^3$, by Lemma 2.4, we have a new spatial embedding $f'_m \in \text{SE}(K_5)$ from f_m by delta moves and ambient isotopies such that $f'_m|_\gamma$ is ambient isotopic to ϕ_γ for any $\gamma \in C^0 \cup C^1 \cup \dots \cup C^6$. Set $C^7 = \{\gamma_0\}$. By claim, we have that $a_2(f'_m(\gamma_0)) = a_2(\phi_{\gamma_0}(\gamma_0))$. By Theorem 2.2, ϕ_{γ_0} is obtained from $f'_m|_{\gamma_0}$ by clasp-pass moves and ambient isotopies. Let $\{e^1, e^2, e^3, e^4\}$ be a

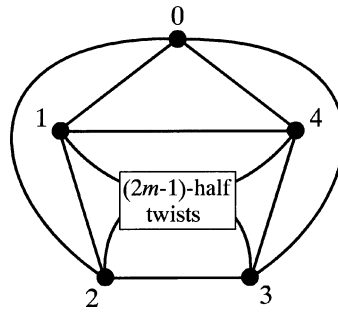


Fig. 7.

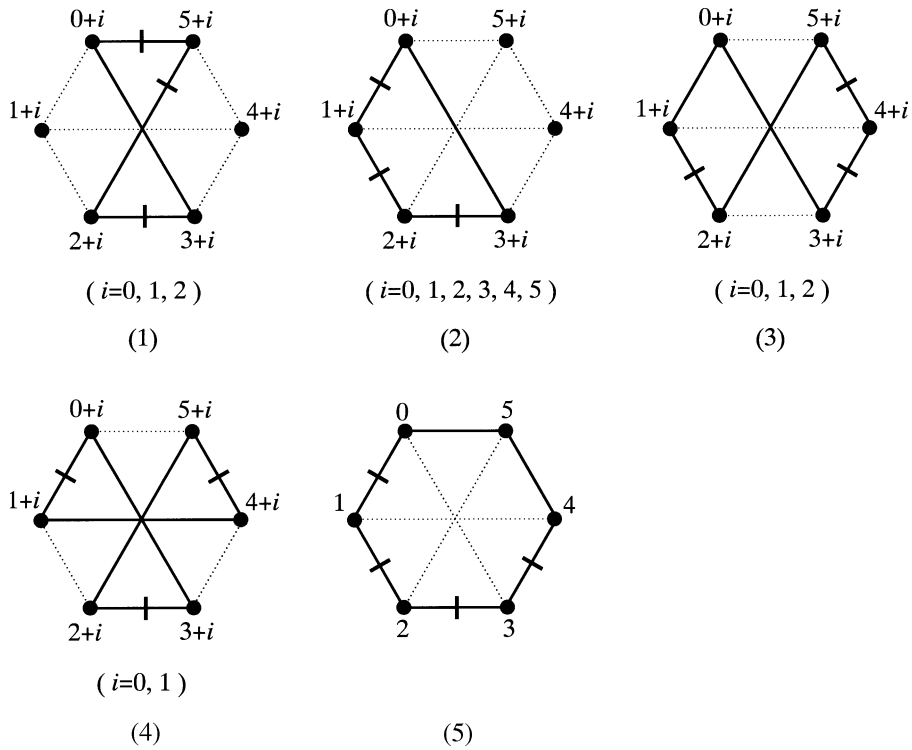


Fig. 8.

set of edges of γ_0 checked in Fig. 6(7). Since γ_0 is a unique cycle in $C^0 \cup C^1 \cup \dots \cup C^7$ containing $e^1 \cup e^2 \cup e^3 \cup e^4$ and the clasp-pass move is a 4-component Brunnian local move, by Lemma 2.3, we have a required spatial embedding.

In the case that $G = K_{3,3}$, we assign the labels $0, 1, 2, 3, 4, 5 \in \mathbb{Z}/6\mathbb{Z}$ to the vertices of $K_{3,3}$ as in Fig. 1(2). We classify the cycles of $K_{3,3}$ into the following six classes; the set of 3-cycles C^0 , the set of cycles C^k as in Fig. 8(k) ($k = 1, 2, \dots, 5$). Note that for checked

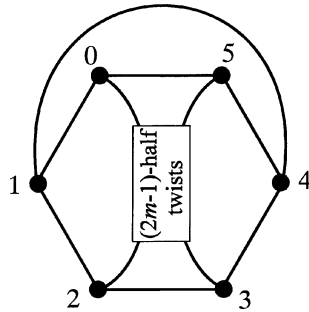


Fig. 9.

edges in Fig. 8(k), there is a unique cycle in $C^0 \cup C^1 \cup \dots \cup C^k$ ($k = 1, 2, \dots, 5$) containing the checked edges. For a spatial embedding $f \in \text{SE}(K_{3,3})$, we set

$$\alpha(f) = \sum_{\gamma \in C_2} a_2(f(\gamma)) - \sum_{\gamma \in C_1} a_2(f(\gamma)).$$

For an integer m , let $f_m \in \text{SE}(K_{3,3})$ be a spatial embedding as in Fig. 9. It is not hard to see that $\alpha(f_m) = m(m - 1)/2$.

Claim. For two spatial embeddings $f, g \in \text{SE}(K_{3,3})$, if f is obtained from g by delta moves, then $\alpha(f) = \alpha(g)$.

Proof. Suppose that f is obtained from g by delta moves. Then by Theorem 4.1, f and g are homologous. In [8], Motohashi and the first author showed that α is a vertex homotopy invariant and that any spatial embedding in $\text{SE}(K_{3,3})$ is vertex homotopic to some spatial embedding in $\{f_m \mid m \in \mathbb{Z}\}$. Thus f and g are vertex homotopic to some f_p and f_q ($p, q \in \mathbb{Z}$) respectively, $\alpha(f) = \alpha(f_p) = p(p - 1)/2$ and $\alpha(g) = \alpha(f_q) = q(q - 1)/2$. Since vertex homotopic spatial embeddings are homologous [16], f_p and f_q are homologous. On the other hand, the first author showed that f_p and f_q are homologous if and only if $p = q$ [18, Example 3.4]. Hence we have $\alpha(f) = \alpha(g)$. \square

By the arguments similar to that in the the case $G = K_5$, we have a required spatial embedding in $\text{SE}(K_{3,3})$ from f_m . \square

3. Proof of Theorem 1.2

Proposition 3.1. Let G be one of the seven graphs in Figs. 2(1), (2), ..., (7). Let $\{\phi_\gamma \in \text{SE}(\gamma) \mid \gamma \in \Gamma_2(G)\}$ be a set of embeddings. If $\sum_{\gamma \in \Gamma_2(G)} \text{lk}_2(\phi_\gamma(\gamma)) \equiv 1 \pmod{2}$, then there is a spatial embedding $f \in \text{SE}(G)$ such that $\text{lk}(f(\gamma)) = \text{lk}(\phi_\gamma(\gamma))$ for any $\gamma \in \Gamma_2(G)$.

Proof. In the case that $G = K_6$, we assign the numbers 0, 1, 2, ..., 5 to the vertices of K_6 as in Fig. 10. Then there are ten pairs of disjoint cycles $\gamma_{ij} = 0ij \cup klm$ ($\{i, j, k, l, m\} =$

$\{1, 2, 3, 4, 5\}$). Let $X = (x_1, x_2, \dots, x_{10})$ and $f_X \in \text{SE}(K_6)$ the spatial embedding as in Fig. 10. Then we have

$$\begin{aligned} \text{lk}(f_X(\gamma_{12})) &= \varepsilon_1 x_1, & \text{lk}(f_X(\gamma_{23})) &= \varepsilon_2 x_2, & \text{lk}(f_X(\gamma_{34})) &= \varepsilon_3 x_3, \\ \text{lk}(f_X(\gamma_{45})) &= \varepsilon_4 x_4, & \text{lk}(f_X(\gamma_{51})) &= \varepsilon_5 x_5, & \text{lk}(f_X(\gamma_{25})) &= \varepsilon_6(x_6 + x_{10} - x_3 + 1), \\ \text{lk}(f_X(\gamma_{31})) &= \varepsilon_7(x_7 + x_6 - x_4 + 1), & \text{lk}(f_X(\gamma_{42})) &= \varepsilon_8(x_8 + x_7 - x_5 + 1), \\ \text{lk}(f_X(\gamma_{53})) &= \varepsilon_9(x_9 + x_8 - x_1 + 1), & \text{lk}(f_X(\gamma_{14})) &= \varepsilon_{10}(x_{10} + x_9 - x_2 + 1), \end{aligned}$$

where $\varepsilon_i = 1$ or -1 ($i = 1, 2, \dots, 10$). Set

$$\begin{aligned} \text{lk}(\phi_{\gamma_{12}}(\gamma_{12})) &= m_1, & \text{lk}(\phi_{\gamma_{23}}(\gamma_{23})) &= m_2, & \text{lk}(\phi_{\gamma_{34}}(\gamma_{34})) &= m_3, \\ \text{lk}(\phi_{\gamma_{45}}(\gamma_{45})) &= m_4, & \text{lk}(\phi_{\gamma_{51}}(\gamma_{51})) &= m_5, & \text{lk}(\phi_{\gamma_{25}}(\gamma_{25})) &= m_6, \\ \text{lk}(\phi_{\gamma_{31}}(\gamma_{31})) &= m_7, & \text{lk}(\phi_{\gamma_{42}}(\gamma_{42})) &= m_8, & \text{lk}(\phi_{\gamma_{53}}(\gamma_{53})) &= m_9, \\ \text{lk}(\phi_{\gamma_{14}}(\gamma_{14})) &= m_{10}. \end{aligned}$$

By setting $\text{lk}(f_X(\gamma_{ij})) = \text{lk}(\phi_{\gamma_{ij}}(\gamma_{ij}))$ ($1 \leq i < j \leq 5$), we have the following system of linear equations

$$\left\{ \begin{array}{l} \varepsilon_1 x_1 = m_1, \\ \varepsilon_2 x_2 = m_2, \\ \varepsilon_3 x_3 = m_3, \\ \varepsilon_4 x_4 = m_4, \\ \varepsilon_5 x_5 = m_5, \\ \varepsilon_6(x_6 + x_{10} - x_3 + 1) = m_6, \\ \varepsilon_7(x_7 + x_6 - x_4 + 1) = m_7, \\ \varepsilon_8(x_8 + x_7 - x_5 + 1) = m_8, \\ \varepsilon_9(x_9 + x_8 - x_1 + 1) = m_9, \\ \varepsilon_{10}(x_{10} + x_9 - x_2 + 1) = m_{10}. \end{array} \right.$$

We can easily see that the system of linear equations above has an integral solution if and only if $\sum_{i=1}^{10} m_i \equiv 1 \pmod{2}$. Thus we have a solution $X = A$ of this equations. The spatial embedding f_A is a required spatial embedding.

By considering the spatial graphs as in Fig. 11 and arguments similar to that in the case $G = K_6$, we can prove the other cases. \square

Proposition 3.2 (Murakami and Nakanishi [9]). *Let L_1 and L_2 be oriented, ordered 2-component links. Then L_1 is obtained from L_2 by delta moves and ambient isotopies if and only if $\text{lk}(L_1) = \text{lk}(L_2)$.*

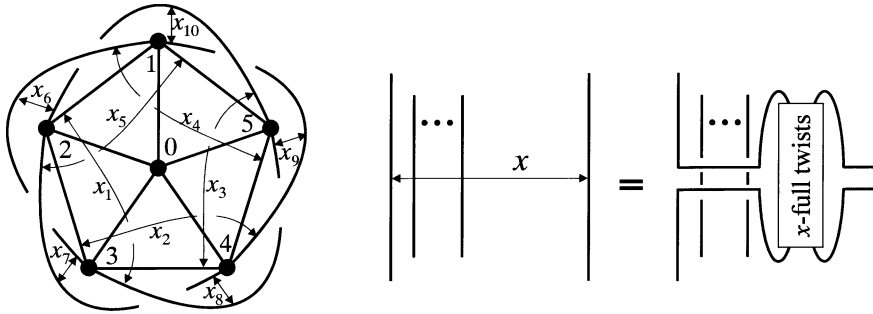


Fig. 10.

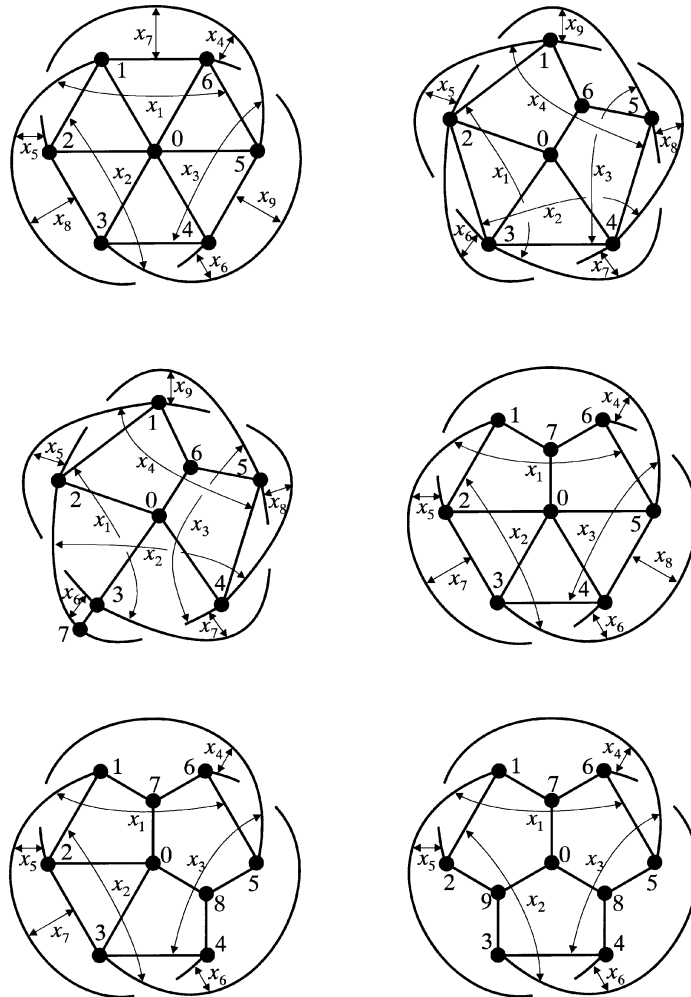


Fig. 11.

By the arguments similar to that in the proof of Lemma 2.3, we have the following lemma.

Lemma 3.3. *Let G be a graph and $\gamma_0 = \alpha_1 \cup \alpha_2$ a pair of disjoint cycles of G . For a spatial embedding $f \in \text{SE}(G)$ and a 2-component link $L = K_1 \cup K_2$, suppose that L is obtained from $f|\gamma_0 = f|\alpha_1 \cup \alpha_2$ by an n -component Brunnian local move (T, S) with an orientation preserving embedding $\varphi: B^3 \rightarrow S^3$. Let e_1, e_2, \dots, e_n be edges in γ_0 (possibly $e_i = e_j$, $i \neq j$) with $e_1 \cup e_2 \cup \dots \cup e_k \subset \alpha_1$ and $e_{k+1} \cup e_{k+2} \cup \dots \cup e_n \subset \alpha_2$, where k is the number of the components of $\varphi(T) \cap f(\alpha_1)$. Then there is a spatial embedding $g \in \text{SE}(G)$ such that*

- (i) $g|\gamma_0 = g|\alpha_1 \cup \alpha_2$ is ambient isotopic to $L = K_1 \cup K_2$,
- (ii) $g|\gamma$ is ambient isotopic to $f|\gamma$ if a pair of disjoint cycles γ does not contain $\bigcup e_i$, and
- (iii) g is obtained from f by an n -component Brunnian local move and ambient isotopies.

Since the delta move is a 3-component Brunnian local move, by Proposition 3.2 and Lemma 3.3, we have the following lemma.

Lemma 3.4. *Let G be a graph and C^1, C^2, \dots, C^n sets of pairs of disjoint cycles of G with $C^i \cap C^j = \emptyset$ ($i \neq j$). Set $C^1 \cup C^2 \cup \dots \cup C^n = \{\alpha_{11} \cup \alpha_{12}, \alpha_{21} \cup \alpha_{22}, \dots, \alpha_{m1} \cup \alpha_{m2}\}$. Suppose that for any k ($= 1, 2, \dots, n$) and for any $\alpha_1 \cup \alpha_2 \in C^k$, there are sets of three edges*

$$\{e_{11}, e_{12}, e_{13}\}, \quad \{e_{21}, e_{22}, e_{23}\}, \quad \{e_{31}, e_{32}, e_{33}\} \quad \text{and} \quad \{e_{41}, e_{42}, e_{43}\}$$

(possibly $e_{pq} = e_{rs}$ ($p, q \neq r, s$)) such that

- (i) for each i ($= 1, 2, 3, 4$) $\alpha_1 \cup \alpha_2$ is a unique element in $C^1 \cup C^2 \cup \dots \cup C^k$ that contains $e_{i1} \cup e_{i2} \cup e_{i3}$,
- (ii) $e_{j1} \cup e_{j2} \cup e_{j3} \subset \alpha_j$ ($j = 1, 2$),
- (iii) $e_{31} \cup e_{32} \subset \alpha_1$ and $e_{33} \subset \alpha_2$, and
- (iv) $e_{41} \subset \alpha_1$ and $e_{42} \cup e_{43} \subset \alpha_2$.

For any spatial embedding $f \in \text{SE}(G)$ and for any 2-component links L_1, L_2, \dots, L_m , if $\text{lk}(f(\alpha_{i1} \cup \alpha_{i2})) = \text{lk}(L_i)$ ($i = 1, 2, \dots, m$), then there is a spatial embedding $g \in \text{SE}(G)$ such that

- (i) $g|\alpha_{i1} \cup \alpha_{i2}$ is ambient isotopic to L_i ($i = 1, 2, \dots, m$) and
- (ii) g is obtained from f by delta moves and ambient isotopies.

Proof of Theorem 1.2. The ‘only if’ part in the case $G = K_6$ is shown in [1, Proof of Theorem 1], and those in the other cases are shown by the arguments similar to that in [1, Proof of Theorem 1].

By combining Proposition 3.1 and Lemma 3.4, we have the ‘if’ part. \square

4. Graph-homology invariant

In this section, we shall introduce a graph-homology invariant that is needed to prove Theorems 1.4 and 1.5.

The spatial-graph homology is defined by the first author as follows [16]. Two spatial embeddings $f, g \in \text{SE}(G)$ ($k = 1, 2$) are said to be *homologous* if there is a locally flat embedding $\Phi : (G \times I) \# \bigcup_{i=1}^n F_i \rightarrow S^3 \times I$ between f and g , where $I = [0, 1]$ is a closed interval, $\{F_i\}_{i=1}^n$ is a finite set of mutually disjoint, closed, orientable surfaces, and $\#$ denotes the connected sum. More precisely, for each F_i , there is just one edge e in G such that F_i is attached to the interior of $e \times I$ by the usual connected sum of surfaces. In [18] the first author gave a homology classification of spatial graphs. An another homology classification was given by the second author [25]. Motohashi and the first author [8] gave a necessary and sufficient condition for spatial graphs to be homologous by using delta move as follows.

Theorem 4.1 (Motohashi and Taniyama [8]). *Two spatial embeddings $f, g \in \text{SE}(G)$ are homologous if and only if f is obtained from g by delta moves and ambient isotopies.*

By the definition of the homologous, we have the following lemma.

Lemma 4.2. *Let G be a graph and H a subgraph of G . Let f and g be spatial embeddings of G . If f and g are homologous, then $f|_H$ and $g|_H$ are homologous.*

Let G be a graph and e an edge of G that is not a loop. Let $f \in \text{SE}(G)$ be a spatial embedding. Then there is a spatial embedding $f' \in \text{SE}(G/e)$ such that $f'(G/e)$ is obtained from $f(G)$ by contracting $f(e)$ into one point, i.e., $f'(G/e) = f(G)/f(e) \subset S^3/f(e) \cong S^3$. Note that f' is unique up to ambient isotopy. We say that f' is *naturally induced* from f . Similarly, for a minor graph H of G , we also define a spatial embedding $f' \in \text{SE}(H)$ to be *naturally induced* from f .

Lemma 4.3. *Let G be a graph and e an edge of G that is not a loop. Let f and g be spatial embeddings of G , and let f' and g' be spatial embeddings of G/e naturally induced from f and g , respectively. If f and g are homologous, then f' and g' are homologous.*

Proof. By [25, Proof of Theorem 1.1], we have the following claim.

Claim. *Let G be a graph and T a maximal subgraph of G that does not contain cycles. Then for any homologous spatial embeddings $f, g \in \text{SE}(G)$, there is a locally flat embedding*

$$\Phi : (G \times I) \# \bigcup_{i=1}^n F_i \rightarrow S^3 \times I$$

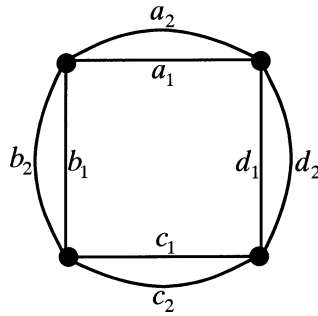


Fig. 12.

such that

$$\Phi\left((T \times I) \# \bigcup_{i=1}^n F_i\right) = \Phi(T \times I) = \Phi(T \times \{0\}) \times I \subset S^3 \times I \quad \text{and} \quad \Phi(G \times \{0\})$$

(respectively $\Phi(G \times \{1\})$) is ambient isotopic to f (respectively g).

If an edge e is not a loop, then there is a maximal subgraph T of G that does not contain cycles such that T contains e . Hence, we obtain from Φ a locally flat embedding $\Phi' : (G/e \times I) \# \bigcup_{i=1}^n F_i \rightarrow S^3 \times I$ between f' and g' . This completes the proof. \square

Let D_4 be a graph as in Fig. 12. Let $\tau : E(D_4) \rightarrow \{0, 1\}$ be a function on the set $E(D_4)$ of all edges of D_4 such that $\{\tau(a_1), \tau(a_2)\} = \{\tau(b_1), \tau(b_2)\} = \{\tau(c_1), \tau(c_2)\} = \{\tau(d_1), \tau(d_2)\} = \{0, 1\}$. We call the function τ a *label function* on $E(D_4)$. Set

$$C_0 = \{a_i b_j c_k d_l \mid \tau(a_i) + \tau(b_j) + \tau(c_k) + \tau(d_l) \text{ is even}\},$$

and

$$C_1 = \{a_i b_j c_k d_l \mid \tau(a_i) + \tau(b_j) + \tau(c_k) + \tau(d_l) \text{ is odd}\}.$$

We define a function $\lambda_\tau : \text{SE}(D_4) \rightarrow \mathbb{Z}$ as follows: For a spatial embedding $f \in \text{SE}(D_4)$, set

$$\lambda_\tau(f) = \sum_{\gamma \in C_0} a_2(f(\gamma)) - \sum_{\gamma \in C_1} a_2(f(\gamma)).$$

Then we have the following proposition.

Proposition 4.4. *If two spatial embeddings $f, g \in \text{SE}(D_4)$ are homologous, then $\lambda_\tau(f) = \lambda_\tau(g)$ for any label function τ .*

Proof. Motohashi and the first author defined a certain property, ‘almost pseudo adjacent’, of graphs [8]. They showed that for an almost pseudo adjacent graph G , two spatial embeddings f and g in $\text{SE}(G)$ are vertex homotopic if and only if f and g are homologous. Since by [8, Theorem 4.2] D_4 is almost pseudo adjacent and by [17, Theorem 1.4] λ_τ is a vertex homotopy invariant, we have λ_τ is a graph-homology invariant. \square

Combining Lemmas 4.2, 4.3 and Proposition 4.4, we have the following theorem.

Theorem 4.5. *Let G be a graph with $D_4 \leq G$. Let $f, g \in \text{SE}(G)$ be spatial embeddings and $f', g' \in \text{SE}(D_4)$ spatial embeddings naturally induced from f, g , respectively. If f and g are homologous, then $\lambda_\tau(f') = \lambda_\tau(g')$ for any label function τ .*

5. Proof of Theorem 1.5

The following four lemmas are useful to find adaptable graphs. By definition of adaptability, the first two lemmas clearly hold. The last two lemmas are slight variations of Lemma 2.4.

Lemma 5.1. *Let G be a graph and e a loop of G . Then G is adaptable if and only if $G - \{e\}$ is adaptable.*

Lemma 5.2. *Let G be a graph. Let e be an edge of G that is not a loop and that has an end vertex of degree 2. Then G is adaptable if and only if G/e is adaptable.*

Lemma 5.3. *Let G be a graph and e an edge of G . Let C be the set of cycles of G containing e . Suppose that C is represented by a disjoint union $C^1 \cup C^2 \cup \dots \cup C^n$ of sets of cycles with the following property: For any $k (= 1, 2, \dots, n)$ and for any $\gamma \in C^k$, there are three edges e_1, e_2 and e_3 (possibly $e_i = e_j, i \neq j$) of G such that γ is a unique cycle in $(\Gamma_1(G) \setminus C) \cup C^1 \cup C^2 \cup \dots \cup C^k$ that contains $e_1 \cup e_2 \cup e_3$. Then G is adaptable if $G - \{e\}$ is adaptable.*

Lemma 5.4. *Let G be a graph and e an edge of G that is not a loop. Let C be the set of cycles of $G - \{e\}$ containing both of the end vertices of e . Suppose that C is represented by a disjoint union $C^1 \cup C^2 \cup \dots \cup C^n$ of sets of cycles with the following property: For any $k (= 1, 2, \dots, n)$ and for any $\gamma \in C^k$, there are three edges e_1, e_2 and e_3 (possibly $e_i = e_j, i \neq j$) of G such that γ is a unique cycle in $(\Gamma_1(G) \setminus C) \cup C^1 \cup C^2 \cup \dots \cup C^k$ that contains $e_1 \cup e_2 \cup e_3$. Then G is adaptable if G/e is adaptable.*

Proof of Theorem 1.5. Let H_1 and H_2 be graphs as in Fig. 13(1), (2), respectively. If H_1 (respectively H_2) is adaptable, then by using Lemma 5.3 repeatedly, $H(l, m, n)$ (respectively H_1) is adaptable. Hence it is sufficient to show that H_2 is adaptable.

Let $f \in \text{SE}(H_2)$ be a spatial embedding as in Fig. 14. Let D_4^{ij} be a subgraph of H_2 as in Fig. 15 and let τ_{ij} be a label function on $E(D_4^{ij})$ with $\tau_{ij}(a_1) = \tau_{ij}(b_1) = \tau_{ij}(c_1) = \tau_{ij}(d_1) = 0$ and $\tau_{ij}(a_i) = \tau_{ij}(b_2) = \tau_{ij}(c_j) = \tau_{ij}(d_2) = 1$. Since $f(D_4^{ij})$ contains only one nontrivial knot that is a x_{ij} -twist knot we have $\lambda_{\tau_{ij}}(f|D_4^{ij}) = x_{ij}$ ($2 \leq i \leq m, 2 \leq j \leq n$). We classify the cycles of H_2 into the following six classes; the set C^0 of 2-cycles,

$$C^1 = \{a_1 b_q c_r d_s \mid 1 \leq q \leq 2, 1 \leq r \leq n, 1 \leq s \leq 2\},$$

$$C^2 = \{a_p b_1 c_r d_s \mid 2 \leq p \leq m, 1 \leq r \leq n, 1 \leq s \leq 2\},$$

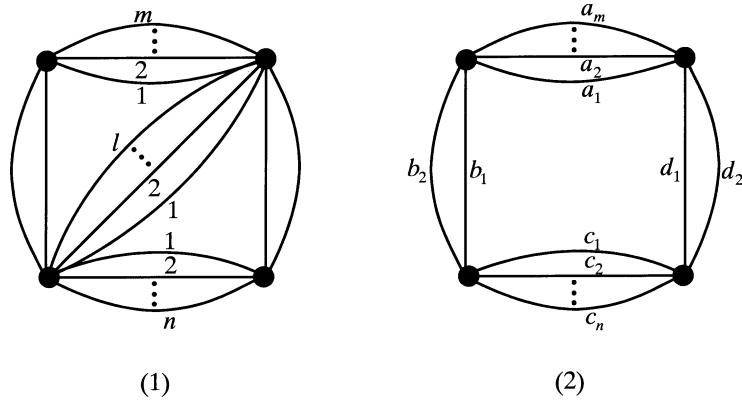


Fig. 13.

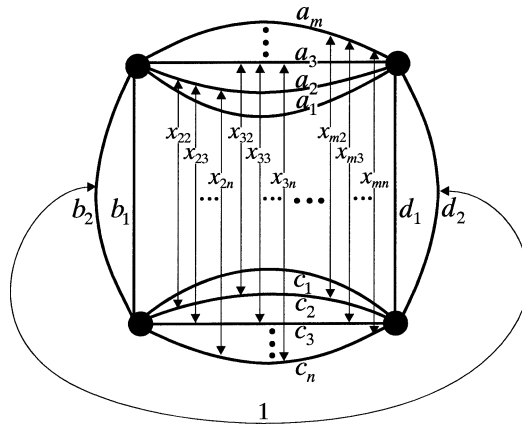


Fig. 14.

$$C^3 = \{a_p b_2 c_1 d_s \mid 2 \leq p \leq m, 1 \leq s \leq 2\},$$

$$C^4 = \{a_p b_2 c_r d_1 \mid 2 \leq p \leq m, 2 \leq r \leq n\}, \quad \text{and}$$

$$C^5 = \{a_p b_2 c_r d_2 \mid 2 \leq p \leq m, 2 \leq r \leq n\}.$$

We note that

- (i) $a_1 b_q c_r d_s \in C^1$ is a unique cycle in C^1 that contains $b_q \cup c_r \cup d_s$ ($1 \leq q \leq 2, 1 \leq r \leq n, 1 \leq s \leq 2$),
- (ii) $a_p b_1 c_r d_s \in C^2$ is a unique cycle in $C^1 \cup C^2$ that contains $a_p \cup c_r \cup d_s$ ($2 \leq p \leq m, 1 \leq r \leq n, 1 \leq s \leq 2$),
- (iii) $a_p b_2 c_1 d_s \in C^3$ is a unique cycle in $C^1 \cup C^2 \cup C^3$ that contains $a_p \cup b_2 \cup d_s$ ($2 \leq p \leq m, 1 \leq s \leq 2$), and
- (iv) $a_p b_2 c_r d_1 \in C^4$ is a unique cycle in $C^1 \cup C^2 \cup C^3 \cup C^4$ that contains $a_p \cup b_2 \cup c_r$ ($2 \leq p \leq m, 2 \leq r \leq n$).

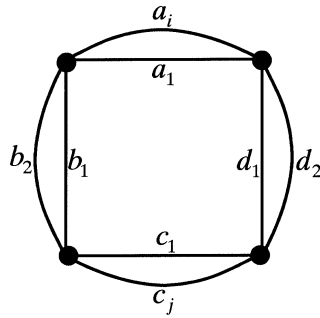


Fig. 15.

By Lemma 2.4, for any set $\{\phi_\gamma \mid \gamma \in \Gamma_1(H_2)\}$ of embeddings, there is a spatial embedding $f' \in \text{SE}(H_2)$ such that

- (i) $f'|_\gamma$ is ambient isotopic to ϕ_γ for any $\gamma \in C^0 \cup C^1 \cup \dots \cup C^4$ and
- (ii) f' is obtained from f by delta moves and ambient isotopies.

By Theorem 4.1, we have f' and f are homologous. Hence by Theorem 4.5, $\lambda_{\tau_{ij}}(f'|_{D_4^{ij}}) = \lambda_{\tau_{ij}}(f|_{D_4^{ij}}) = x_{ij}$ ($2 \leq i \leq m, 2 \leq j \leq n$). Note that each $a_p b_2 c_r d_2 \in C^5$ is contained in D_4^{pr} , and that it is not contained in D_4^{ij} if $(i, j) \neq (p, r)$. Other fifteen cycles in D_4^{pr} are contained in $C^1 \cup C^2 \cup C^3 \cup C^4$. Therefore by an appropriate choice of x_{pr} we may assume that $a_2(f'(\gamma)) = a_2(\phi_\gamma(\gamma))$ for any $\gamma \in C^5$. Since any cycle in H_2 is determined by at most four edges, by Theorem 2.2 and Lemma 2.3, we have a required spatial embedding from f' . \square

6. Proof of Theorem 1.4

Theorem 6.1 (Soma et al. [14]). *Let G be a (connected) graph and $h_0 : G \rightarrow S^2$ a planar embedding. Let $\{D_1, D_2, \dots, D_n\}$ be the set of closures of the connected components of $S^2 - h_0(G)$. Then two spatial embeddings $f, g \in \text{SE}(G)$ are homologous if and only if*

$$\text{lk}(f(h_0^{-1}(\partial D_i)), f(h_0^{-1}(\partial D_j))) = \text{lk}(g(h_0^{-1}(\partial D_i)), g(h_0^{-1}(\partial D_j)))$$

for all disjoint pairs $\partial D_i, \partial D_j$.

Proposition 6.2. *Let G_i be a graph as in Fig. 3(i) ($i = 1, 2, \dots, 8$). Then a spatial embedding $f \in \text{SE}(G_i)$ is homologous to a spatial embedding as in Fig. 16(i).*

Let τ be a label function on $E(D_4)$. We define a function $\lambda : \text{SE}(D_4) \rightarrow \mathbb{Z}$ by $\lambda(f) = |\lambda_\tau(f)|$ for any $f \in \text{SE}(D_4)$. Note that λ does not depend on a choice of a label function. By Proposition 4.4 and Theorem 6.1 we have that $\lambda(f)$ is determined by the linking numbers $\text{lk}(f(a_1 a_2), f(c_1 c_2))$ and $\text{lk}(f(b_1 b_2), f(d_1 d_2))$. In fact it is not hard to check that $\lambda(f) = |\text{lk}(f(a_1 a_2), f(c_1 c_2)) \text{lk}(f(b_1 b_2), f(d_1 d_2))|$. Then by counting the linking numbers we have the following eight propositions.

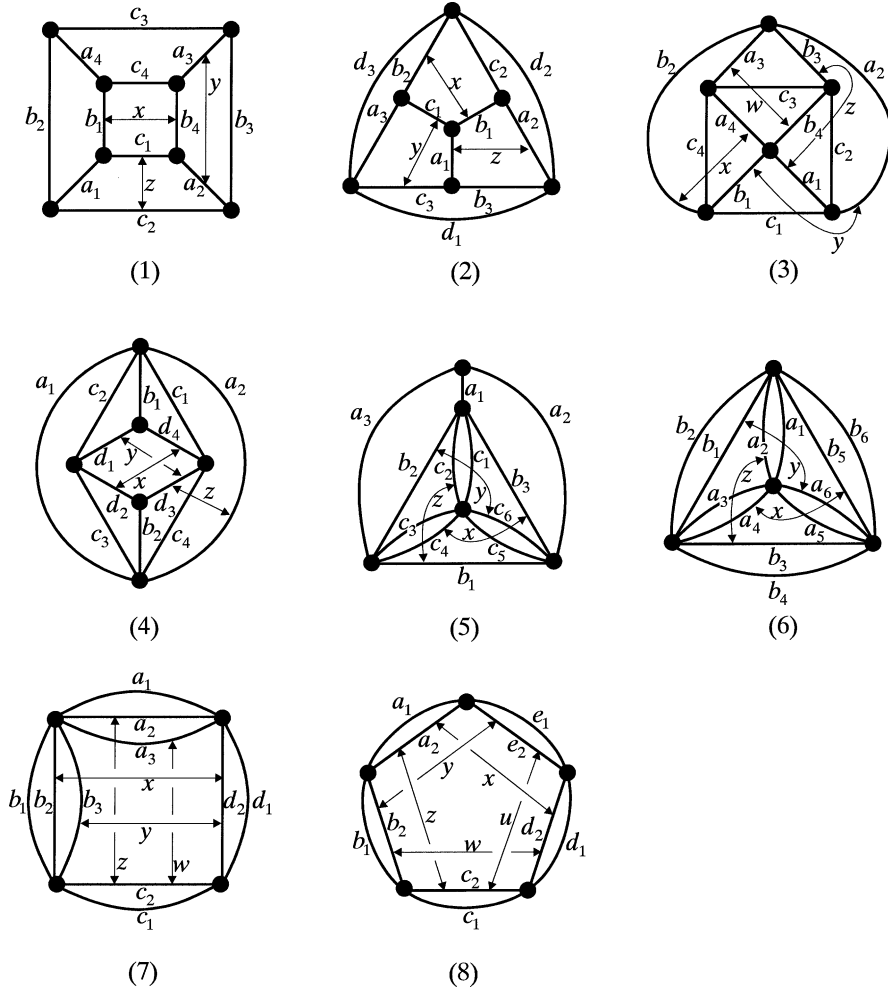


Fig. 16.

Proposition 6.3. *Let $f \in \text{SE}(G_1)$ be a spatial embedding as in Fig. 16(1). Let $D_4^1 = (((G_1/a_1)/a_2)/a_3)/a_4$, $D_4^2 = (((G_1/b_1)/b_2)/b_3)/b_4$, $D_4^3 = (((G_1/c_1)/c_2)/c_3)/c_4$, and let $f_i \in \text{SE}(D_4^i)$ ($i = 1, 2, 3$) be spatial embeddings naturally induced from f . Then we have $\lambda(f_1) = |xy|$, $\lambda(f_2) = |xz|$, and $\lambda(f_3) = |yz|$.*

Proposition 6.4. *Let $f \in \text{SE}(G_2)$ be a spatial embedding as in Fig. 16(2). Let $D_4^1 = (((G_2 - d_1)/a_1)/a_2)/a_3$, $D_4^2 = (((G_2 - d_2)/b_1)/b_2)/b_3$, $D_4^3 = (((G_2 - d_3)/c_1)/c_2)/c_3$, and let $f_i \in \text{SE}(D_4^i)$ ($i = 1, 2, 3$) be spatial embeddings naturally induced from f . Then we have $\lambda(f_1) = |xz|$, $\lambda(f_2) = |xy|$, and $\lambda(f_3) = |yz|$.*

Proposition 6.5. *Let $f \in \text{SE}(G_3)$ be a spatial embedding as in Fig. 16(3). Let $D_4^1 = ((G_3 - a_1 \cup a_3)/a_2)/a_4$, $D_4^2 = ((G_3 - a_2 \cup a_4)/a_1)/a_3$, $D_4^3 = ((G_3 - b_1 \cup b_3)/b_2)/b_4$,*

$D_4^4 = ((G_3 - b_2 \cup b_4)/b_1)/b_3$, and let $f_i \in \text{SE}(D_4^i)$ ($i = 1, 2, 3, 4$) be spatial embeddings naturally induced from f . Then we have $\lambda(f_1) = |xy|$, $\lambda(f_2) = |zw|$, $\lambda(f_3) = |xw|$ and $\lambda(f_4) = |zy|$.

Proposition 6.6. *Let $f \in \text{SE}(G_4)$ be a spatial embedding as in Fig. 16(4). Let $D_4^1 = ((G_4 - a_1 \cup a_2)/b_1)/b_2$, $D_4^2 = ((G_4 - c_1 \cup c_3)/d_1)/d_3$, $D_4^3 = ((G_4 - c_2 \cup c_4)/d_2)/d_4$, and let $f_i \in \text{SE}(D_4^i)$ ($i = 1, 2, 3$) be spatial embeddings naturally induced from f . Then we have $\lambda(f_1) = |xy|$, $\lambda(f_2) = |yz|$, and $\lambda(f_3) = |xz|$.*

Proposition 6.7. *Let $f \in \text{SE}(G_5)$ be a spatial embedding as in Fig. 16(5). Let $D_4^1 = (G_5 - b_1 \cup c_1 \cup c_2)/a_1$, $D_4^2 = (G_5 - b_2 \cup c_5 \cup c_6)/a_2$, $D_4^3 = (G_5 - b_3 \cup c_3 \cup c_4)/a_3$, and let $f_i \in \text{SE}(D_4^i)$ ($i = 1, 2, 3$) be spatial embeddings naturally induced from f . Then we have $\lambda(f_1) = |xy|$, $\lambda(f_2) = |xz|$, and $\lambda(f_3) = |yz|$.*

Proposition 6.8. *Let $f \in \text{SE}(G_6)$ be a spatial embedding as in Fig. 16(6). Let $D_4^1 = G_6 - a_1 \cup a_2 \cup b_3 \cup b_4$, $D_4^2 = G_6 - a_3 \cup a_4 \cup b_5 \cup b_6$, $D_4^3 = G_6 - a_5 \cup a_6 \cup b_1 \cup b_2$, and let $f_i \in \text{SE}(D_4^i)$ ($i = 1, 2, 3$) be spatial embeddings naturally induced from f . Then we have $\lambda(f_1) = |xy|$, $\lambda(f_2) = |yz|$, and $\lambda(f_3) = |xz|$.*

Proposition 6.9. *Let $f \in \text{SE}(G_7)$ be a spatial embedding as in Fig. 16(7). Let $D_4^1 = G_7 - a_3 \cup b_3$, $D_4^2 = G_7 - a_3 \cup b_2$, $D_4^3 = G_7 - a_2 \cup b_3$, $D_4^4 = G_7 - a_2 \cup b_2$, and let $f_i \in \text{SE}(D_4^i)$ ($i = 1, 2, 3, 4$) be spatial embeddings naturally induced from f . Then we have $\lambda(f_1) = |xz|$, $\lambda(f_2) = |yz|$, $\lambda(f_3) = |xw|$ and $\lambda(f_4) = |yw|$.*

Proposition 6.10. *Let $f \in \text{SE}(G_8)$ be a spatial embedding as in Fig. 16(8). Let $D_4^1 = (G_8 - a_2)/a_1$, $D_4^2 = (G_8 - b_2)/b_1$, $D_4^3 = (G_8 - c_2)/c_1$, $D_4^4 = (G_8 - d_2)/d_1$, $D_4^5 = (G_8 - e_2)/e_1$, and let $f_i \in \text{SE}(D_4^i)$ ($i = 1, 2, 3, 4, 5$) be spatial embeddings naturally induced from f . Then we have $\lambda(f_1) = |uw|$, $\lambda(f_2) = |xu|$, $\lambda(f_3) = |xy|$, $\lambda(f_4) = |yz|$ and $\lambda(f_5) = |zw|$.*

Theorem 6.11. *The graphs G_i ($i = 1, 2, \dots, 8$) are not adaptable.*

In [11], Ohyama and the first author showed that G_8 is not adaptable. Our proof for G_8 is somewhat different from theirs.

Proof. We show that G_1 is not adaptable. Let $f \in \text{SE}(G_1)$ be a spatial embedding. Let $\gamma_1 = b_1c_1b_4c_4$, $\gamma_2 = a_1c_1a_2c_2$ and $\gamma_3 = a_2b_4a_3b_3$ be cycles in G_1 . We note that γ_i is a 4-cycle of D_4^i ($i = 1, 2, 3$), γ_i is not a 4-cycle in D_4^j if $i \neq j$, and that for each 4-cycle γ' in D_4^i there is a unique cycle γ in G_1 that naturally corresponds to γ' ($i = 1, 2, 3$). Hence, if $a_2(f(\gamma)) = 0$ for any γ ($\neq \gamma_1, \gamma_2$ nor γ_3), then $\lambda(f_i) = |a_2(f(\gamma_i))|$. Combining this fact, Theorem 4.5, Propositions 6.2 and 6.3, we have that if $a_2(f(\gamma)) = 0$ for any γ ($\neq \gamma_1, \gamma_2$ nor γ_3), then there are some integer x, y and z such that $|a_2(f(\gamma_1))| = |xy|$, $|a_2(f(\gamma_2))| = |xz|$ and $|a_2(f(\gamma_3))| = |yz|$. This implies that if $|a_2(f(\gamma_i))|$ ($i = 1, 2, 3$) are

mutually coprime, then $|a_2(f(\gamma_i))| = 1$. On the other hand, for any integer n there exists a knot J with $a_2(K) = n$. It follows that G_1 is not adaptable.

The other cases can be proved in the same way. \square

Theorem 6.12. *Any proper minor of G_i ($i = 1, 2, \dots, 8$) is adaptable.*

Proof. Since a minor graph of an adaptable graph is adaptable, by symmetries of G_i ($i = 1, 2, \dots, 8$), it is sufficient to show that the following 32 graphs, $G_1/a_1, G_1 - a_1, G_2/a_1, G_2 - a_1, G_2/a_2, G_2 - a_2, G_2/d_1, G_2 - d_1, G_3/a_1, G_3 - a_1, G_4/a_1, G_4 - a_1, G_4/b_1, G_4 - b_1, G_4/c_1, G_4 - c_1, G_4/d_1, G_4 - d_1, G_5/a_1, G_5 - a_1, G_5/b_1, G_5 - b_1, G_5/c_1, G_5 - c_1, G_6/a_1, G_6 - a_1, G_7/a_1, G_7 - a_1, G_7/c_1, G_7 - c_1, G_8/a_1, G_8 - a_1$, are adaptable. By combining Theorem 1.5, Lemmas 2.4, 5.1–5.4, it is not hard to see that these graphs are adaptable. In fact, by using Lemmas 5.1–5.4, each graph in these is reduced to a graph that is a minor graph of a graph as Fig. 4 or that can be seen to be adaptable by using Lemma 2.4. \square

Proof of Theorem 1.4. Theorem 1.4 follows directly from Theorems 6.11 and 6.12. \square

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