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C_k-moves on spatial theta-curves and Vassiliev invariants

Akira Yasuhara

Department of Mathematics, Tokyo Gakugei University, Nukuikita 4-1-1, Koganei, Tokyo 184-8501, Japan

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Abstract

The C_k-equivalence is an equivalence relation generated by C_k-moves defined by Habiro. Habiro showed that the set of C_k-equivalence classes of the knots forms an abelian group under the connected sum and it can be classified by the additive Vassiliev invariant of order $\leqslant k-1$. We see that the set of C_k-equivalence classes of the spatial θ-curves forms a group under the vertex connected sum and that if the group is abelian, then it can be classified by the additive Vassiliev invariant of order $\leqslant k-1$. However the group is not necessarily abelian. In fact, we show that it is nonabelian for $k \geqslant 12$. As an easy consequence, we have the set of C_k-equivalence classes of m-string links, which forms a group under the composition, is nonabelian for $k \geqslant 12$ and $m \geqslant 2$.

MSC: primary 57M25; secondary 57M27

Keywords: Spatial theta-curve; C_n-move; Vassiliev invariant; Finite type invariant

1. C_k-moves and Vassiliev invariants of spatial θ-curves

A *tangle* T is a disjoint union of properly embedded arcs in the unit 3-ball B^3. A *local move* is a pair of tangles (T_1, T_2) with $\partial T_1 = \partial T_2$ such that for each component t of T_1 there exists a component u of T_2 with $\partial t = \partial u$. Two local moves (T_1, T_2) and (U_1, U_2) are *equivalent* if there is an orientation preserving self-homeomorphism $\psi : B^3 \to B^3$ such that $\psi(T_i)$ and U_i are ambient isotopic in B^3 relative to ∂B^3 for $i = 1, 2$. Here $\psi(T_i)$ and U_i are *ambient isotopic in* B^3 *relative to* ∂B^3 if $\psi(T_i)$ is deformed to U_i by an ambient isotopy of B^3 that is pointwisely fixed on ∂B^3.

E-mail address: yasuhara@u-gakugei.ac.jp (A. Yasuhara).

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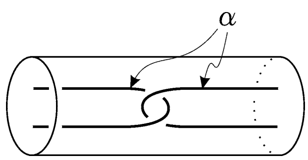

Fig. 1.1.

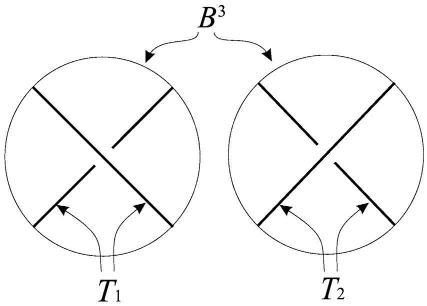

Fig. 1.2.

Let (T_1, T_2) be a local move, t_1 a component of T_1 and t_2 a component of T_2 with $\partial t_1 = \partial t_2$. Let N_1 and N_2 be regular neighbourhoods of t_1 and t_2 in $(B^3 - T_1) \cup t_1$ and $(B^3 - T_2) \cup t_2$ respectively such that $N_1 \cap \partial B^3 = N_2 \cap \partial B^3$. Let α be a disjoint union of properly embedded arcs in $B^2 \times [0, 1]$ as illustrated in Fig. 1.1. Let $\psi_i : B^2 \times [0, 1] \to N_i$ be a homeomorphism with $\psi_i(B^2 \times \{0, 1\}) = N_i \cap \partial B^3$ for $i = 1, 2$. Suppose that $\psi_1(\partial\alpha) = \psi_2(\partial\alpha)$ and $\psi_1(\alpha)$ and $\psi_2(\alpha)$ are ambient isotopic in B^3 relative to ∂B^3. Then we say that a local move $((T_1 - t_1) \cup \psi_1(\alpha), (T_2 - t_2) \cup \psi_2(\alpha))$ is a *double* of (T_1, T_2) with respect to the components t_1 and t_2.

A C_1*-move* is a local move (T_1, T_2) as illustrated in Fig. 1.2. A double of a C_k-move is called a C_{k+1}*-move*. Note that, for each natural number k, there are only finitely many C_k-moves up to equivalence. It is easy to see that if (T_1, T_2) is a C_n-move, then (T_2, T_1) is equivalent to a C_n-move (but possibly not equivalent to itself). The definition of C_k-move follows that in [2], and is different from the one in [3]. However by an easy induction on k it is shown that these two definitions are essentially same. In [3], a C_k-move is called a *simple* C_k-move, and a C_k-move means a *parallel* of a C_k-move. The definition of parallel of a local move appears in Section 3.

Let G be a graph with labeled vertices and edges. Let $f : G \to S^3$ be an embedding of G into the oriented three sphere S^3. The embedding f is called a *spatial graph*.

Let f_1 and f_2 be spatial graphs. We say that f_2 is *obtained from* f_1 *by a local move* (T_1, T_2) if there is an orientation preserving embedding $h : B^3 \to S^3$ such that $f_i(G) \cap h(B^3) = h(T_i)$ for $i = 1, 2$ and $f_1(G - h(B^3)) = f_2(G - h(B^3))$ together with the labels of vertices and edges. Two spatial graphs f_1 and f_2 are C_k*-equivalent* if f_2 is obtained from f_1 by a finite sequence of C_k-moves and ambient isotopies. We note that the relation is an equivalence relation on spatial graphs. For a spatial graph f, let $[f]_k$ denote the C_k-equivalence class that contains f. It is known that C_k-equivalence implies C_{k-1}-equivalence [3,19].

Let l be a positive integer and $k_1, \ldots, k_l$ positive integers. Suppose that for each $P \subset \{1, \ldots, l\}$, an spatial graph f_P in S^3 is assigned. Suppose that there are mutually disjoint, orientation preserving embeddings $h_i : B^3 \to S^3$ $(i = 1, \ldots, l)$ such that

(1) $f_\emptyset(G) - \bigcup_{i=1}^{l} h_i(B^3) = f_P(G) - \bigcup_{i=1}^{l} h_i(B^3)$ together with the labels for any subset $P \subset \{1, \ldots, l\}$,
(2) $(h_i^{-1}(f_\emptyset(G)), h_i^{-1}(f_{\{1,\ldots,l\}}(G)))$ is a C_{k_i}-move $(i = 1, \ldots, l)$, and
(3) $$f_P(G) \cap h_i(B^3) = \begin{cases} f_{\{1,\ldots,l\}}(G) \cap h_i(B^3) & \text{if } i \in P, \\ f_\emptyset(G) \cap h_i(B^3) & \text{otherwise.} \end{cases}$$

Then we call the set $\{f_P \mid P \subset \{1, \ldots, l\}\}$ a *singular spatial graph of type* $(k_1, \ldots, k_l)$.

The θ*-curve* is a graph θ with two vertices v_1, v_2 and three edges e_1, e_2, e_3 each of which joins v_1 and v_2. When $G = \theta$, a spatial graph and singular spatial graph are called *spatial* θ*-curve* and *singular spatial* θ*-curve* respectively. A spatial θ-curve is *trivial* if its image is contained in a 2-sphere in S^3. In the remainder of this section, we consider only the case that a graph is the θ-curve.

Let Γ be the set of all spatial θ-curve types in S^3 and $\mathbb{Z}\Gamma$ the free abelian group generated by Γ. For a singular spatial θ-curve $\boldsymbol{f} = \{f_P \mid P \subset \{1, \ldots, l\}\}$ of type $(k_1, \ldots, k_l)$, we define an element $\kappa(\boldsymbol{f})$ of $\mathbb{Z}\Gamma$ by

$$\kappa(\boldsymbol{f}) = \sum_{P \subset \{1,\ldots,l\}} (-1)^{|P|} f_P.$$

Let $\mathcal{V}(k_1, \ldots, k_l)$ be the subgroup of $\mathbb{Z}\Gamma$ generated by all $\kappa(\boldsymbol{f})$ where $\boldsymbol{f}$ varies over all singular spatial θ-curves of type $(k_1, \ldots, k_l)$.

For two spatial θ-curves f_1 and f_2, remove small balls centered at $f_1(v_2)$ and $f_2(v_1)$ from S^3, then identify the boundaries so that the images of ith edge are joined for each i. Then we obtain a new spatial θ-curve. We call this embedding the *vertex connected sum* of f_1 and f_2, and denote by $f_1 \# f_2$. The vertex connected sum is well-defined up to ambient isotopy [21]. Let $f_1 \# f_2$ be the vertex connected sum of two spatial θ-curves f_1 and f_2. Then $f_1 \# f_2 - f_1 - f_2 \in \mathbb{Z}\Gamma$ is called a *composite relator*. Let $\mathcal{R}_\#$ be the subgroup of $\mathbb{Z}\Gamma$ generated by all composite relators.

Let $\iota : \Gamma \to \mathbb{Z}\Gamma$ be the natural inclusion map. Let $\pi : \mathbb{Z}\Gamma \to \mathbb{Z}\Gamma/\mathcal{V}(k_1, \ldots, k_l)$ and $\lambda : \mathbb{Z}\Gamma/\mathcal{V}(k_1, \ldots, k_l) \to \mathbb{Z}\Gamma/(\mathcal{V}(k_1, \ldots, k_l) + \mathcal{R}_\#)$ be the quotient homomorphisms. Then the composite maps $\pi \circ \iota : \Gamma \to \mathbb{Z}\Gamma/\mathcal{V}(k_1, \ldots, k_l)$ and $\lambda \circ \pi \circ \iota : \Gamma \to \mathbb{Z}\Gamma/(\mathcal{V}(k_1, \ldots, k_l) + \mathcal{R}_\#)$ are called the *universal Vassiliev invariant of type* $(k_1, \ldots, k_l)$ and *universal additive Vassiliev invariant of type* $(k_1, \ldots, k_l)$ respectively. We denote them by $v_{(k_1,\ldots,k_l)}$ and $w_{(k_1,\ldots,k_l)}$ respectively. In the case of knots, these are same invariants as defined by K. Taniyama and the author [20]. Similarly, we can also define $\mathcal{V}(k_1, \ldots, k_l)$ and the universal Vassiliev invariant $v_{(k_1,\ldots,k_l)}$ for the embeddings of any graph. Since a C_1-move is a crossing change we see that a singular spatial graph of type

$$(\underbrace{1, \ldots, 1}_{l})$$

is essentially the same as a singular spatial graph with l crossing vertices in the sense of Stanford [15]. Therefore we see that $v_{(1,\ldots,1)}$ is the universal Vassiliev invariant of order

$\leqslant l-1$. Note that $v_{(1,\ldots,1)}(f_1)=v_{(1,\ldots,1)}(f_2)$ if and only if $v(f_1)=v(f_2)$ for any Vassiliev invariant v of order $\leqslant l-1$. In the case of links, $v_{(2,\ldots,2)}$ is the same as that defined in [8,16]. In [19] Taniyama and the author defined finite type invariants of order $(k;n)$ for the embeddings of a graph, which are essentially same as

$$v_{(\underbrace{n-1,\ldots,n-1}_{k+1})}.$$

By the arguments similar to that in proofs of Theorems 1.1 and 1.2 in [20] and that in proof of Theorem 1.4 in [20], we have the following two theorems.

Theorem 1.1. *Let $k_1,\ldots,k_l$ be positive integers and $k=k_1+\cdots+k_l$. Then the followings hold.*

(1) $\mathcal{V}(k)\subset\mathcal{V}(k_1,\ldots,k_l)\subset\mathcal{V}(\underbrace{1,\ldots,1}_{k})$.

(2) $\mathcal{V}(k_1,\ldots,k_l)+\mathcal{R}_\#=\mathcal{V}(k)+\mathcal{R}_\#$.

Remark. Theorem 1.1(1) holds for the spatial embeddings of any graph.

Theorem 1.2. *The C_k-equivalence classes of the spatial θ-curves forms a group with the unit element $[f_0]_k$ under the vertex connected sum, where f_0 is a trivial θ-curve.*

We denote by G_k this group. Let $\varphi: G_k\to\mathbb{Z}\Gamma/\mathcal{V}(k)$ be a map induced by the inclusion $\iota:\Gamma\to\mathbb{Z}\Gamma$. By Theorem 1.2, φ is a well-defined, epimorphism. (In fact, φ is an isomorphism, see the remark after Corollary 1.4.) Since $\varphi([G_k,G_k])=\mathcal{R}_\#$, by Theorem 1.1(2), we have the following theorem.

Theorem 1.3. *Let $k_1,\ldots,k_l$ be positive integers and $k=k_1+\cdots+k_l$. Then $G_k/[G_k,G_k]$ is isomorphic to $\mathbb{Z}\Gamma/(\mathcal{V}(k_1,\ldots,k_l)+\mathcal{R}_\#)$.*

Let f_1 and f_2 be spatial θ-curves and $k=k_1+\cdots+k_l$. If f_1 and f_2 are C_k-equivalent, then by Theorem 1.1(1), $f_1-f_2\in\mathcal{V}(k)\subset\mathcal{V}(k_1,\ldots,k_l)$. Therefore we have $v_{(k_1,\ldots,k_l)}(f_1)=v_{(k_1,\ldots,k_l)}(f_2)$. On the other hand, if $v_{(k_1,\ldots,k_l)}(f_1)=v_{(k_1,\ldots,k_l)}(f_2)$, then $w_{(k_1,\ldots,k_l)}(f_1)=w_{(k_1,\ldots,k_l)}(f_2)$. Hence we have $f_1-f_2\in\mathcal{V}(k_1,\ldots,k_l)+\mathcal{R}_\#$. If G_k is abelian group, i.e., $[G_k,G_k]=\{id\}$, then by Theorem 1.3, f_1 and f_2 are C_k-equivalent. So we have the following corollary.

Corollary 1.4. *Let $k_1,\ldots,k_l$ be positive integers and $k=k_1+\cdots+k_l$. Let f_1 and f_2 be spatial θ-curves. If G_k is an abelian group, then the following conditions are mutually equivalent.*

(1) *f_1 and f_2 are C_k-equivalent,*
(2) $v_{(k_1,\ldots,k_l)}(f_1)=v_{(k_1,\ldots,k_l)}(f_2)$,
(3) $w_{(k_1,\ldots,k_l)}(f_1)=w_{(k_1,\ldots,k_l)}(f_2)$.

Remark. Let f_1 and f_2 be spatial graphs (not necessarily θ-curve). If $f_1 - f_2 \in \mathcal{V}(k)$, then there are singular spatial graphs $\boldsymbol{f}^i$'s of type (k) and integers x_i's such that $f_1 - f_2 = \sum x_i \kappa(\boldsymbol{f}^i)$. By induction on $\sum |x_i|$, we see that f_1 and f_2 are C_k-equivalent. Since $f_1 - f_2 \in \mathcal{V}(k)$ if f_1 and f_2 are C_k-equivalent, we have the following: Two spatial graphs f_1 and f_2 are C_k-equivalent if and only if $v_{(k)}(f_1) = v_{(k)}(f_2)$.

Theorem 1.5. *Let f_0 be a trivial spatial θ-curve. Then the followings hold.*

(1) *For each $f \in [f_0]_k$, $[f]_{k+2}$ belongs to the center of G_{k+2}.*
(2) *If $2l \geqslant k > l$, then the set $H_k^l = \{[f]_k \mid f \in [f_0]_l\}$ is an abelian subgroup of G_k.*

By [18] and [9], we have $[f_0]_1 = [f_0]_2 = \Gamma$. Hence, by Theorem 1.5(2), $H_4^2 = G_4$ is abelian. By Corollary 1.4, we have

Corollary 1.6. *Let $k_1, \ldots, k_l$ $(l \leqslant 4)$ be positive integers and $k = k_1 + \cdots + k_l \leqslant 4$. Let f_1 and f_2 be spatial θ-curves. Then the following conditions are mutually equivalent.*

(1) *f_1 and f_2 are C_k-equivalent,*
(2) $v_{(k_1,\ldots,k_l)}(f_1) = v_{(k_1,\ldots,k_l)}(f_2)$,
(3) $w_{(k_1,\ldots,k_l)}(f_1) = w_{(k_1,\ldots,k_l)}(f_2)$.

Remark. As a special case of Corollary 1.6, we see that for $k \leqslant 4$ two spatial θ-curves are C_k-equivalent if and only if the universal (additive) Vassiliev invariant of order $\leqslant k-1$ are equal. Meanwhile, a basis for the space of Vassiliev invariants of order $\leqslant 4$ is known [5,7].

Theorem 1.7. *Let f_0 be a trivial spatial θ-curve, and let f_1 and f_2 be in $[f_0]_k$. Then f_1 and f_2 are C_{2k}-equivalent if and only if $v_{(k,k)}(f_1) = v_{(k,k)}(f_2)$.*

As we saw before, G_k is abelian for $k \leqslant 4$. However G_k is not necessarily abelian. In fact, we have the following theorem.

Theorem 1.8. *The group G_k is nonabelian for any $k \geqslant 12$.*

Remarks. (1) If G_k is abelian, then so is $G_{k'}$ for any $k' (< k)$.

(2) In the proof of Theorem 1.8, we see that there are two spatial θ-curves g and h such that $g - h$ is in $\mathcal{R}_\#$ and is not in

$$\mathcal{V}(\underbrace{1, \ldots, 1}_{k}) \quad \text{for } k \geqslant 12.$$

Hence $v_{(k_1,\ldots,k_l)}(g) \neq v_{(k_1,\ldots,k_l)}(h)$ for $k_1 + \cdots + k_l \geqslant 12$ by Theorem 1.1(1), while $w_{(k_1,\ldots,k_l)}(g) = w_{(k_1,\ldots,k_l)}(h)$ for any $k_1, \ldots, k_l$. In contrast, for any knots K and K', $v_{(k_1,\ldots,k_l)}(K) = v_{(k_1,\ldots,k_l)}(K')$ if and only if $w_{(k_1,\ldots,k_l)}(K) = w_{(k_1,\ldots,k_l)}(K')$ [20].

(3) Habiro showed the set of C_k-equivalence classes $S_k(m)$ of m-string links forms a group under the composition [3]. By considering the complement of a regular

neighbourhood of an edge of a spatial θ-curve, we have that there is a surjection from the 2-string links to the spatial θ-curves. Since the surjection induces an epimorphism from $S_k(2)$ to G_k and since there is an epimorphism from $S_k(m)$ $(m > 2)$ to $S_k(2)$, $S_k(m)$ is nonabelian for any $m \geqslant 2$ and $k \geqslant 12$.

The following is still open.

Problem. Find the minimum number k $(5 \leqslant k \leqslant 12)$ such that G_k is nonabelian.

2. Band description of spatial graphs

A C_1-*link model* is a pair (α, β) where α is a disjoint union of properly embedded arcs in B^3 and β is a disjoint union of arcs on ∂B^3 with $\partial\alpha = \partial\beta$ as illustrated in Fig. 2.1. Suppose that a C_k-link model (α, β) is defined where α is a disjoint union of $k+1$ properly embedded arcs in B^3 and β is a disjoint union of $k+1$ arcs on ∂B^3 with $\partial\alpha = \partial\beta$ such that $\alpha \cup \beta$ is a disjoint union of $k+1$ circles. Let γ be a component of $\alpha \cup \beta$ and W a regular neighbourhood of γ in $(B^3 - (\alpha \cup \beta)) \cup \gamma$. Let V be an oriented solid torus, D a disk in ∂V, α_0 properly embedded arcs in V and β_0 arcs on D as illustrated in Fig. 2.2. Let $\psi : V \to W$ be an orientation preserving homeomorphism such that $\psi(D) = W \cap \partial B^3$ and $\psi(\alpha_0 \cup \beta_0)$ bounds disjoint disks in B^3. Then we call the pair $((\alpha - \gamma) \cup \psi(\alpha_0), (\beta - \gamma) \cup \psi(\beta_0))$ a C_{k+1}-*link model.* A *link model* is a C_k-link model for some k. It is known that, for a C_k-link model (α, β), the local move $(\alpha, \hat{\beta})$ is equivalent to a C_k-move [20], where $\hat{\beta}$ is a slight push in of β.

Let f_1 be a spatial θ-curve, and let $(\alpha_1, \beta_1), \ldots, (\alpha_l, \beta_l)$ be link models. Let $\psi_i : B^3 \to S^3$ $(i = 1, \ldots, l)$ be mutually disjoint, orientation preserving embeddings, and let $b_{1,1}, b_{1,2}, \ldots, b_{1,\rho(1)}, b_{2,1}, b_{2,2}, \ldots, b_{2,\rho(2)}, \ldots, b_{l,1}, b_{l,2}, \ldots, b_{l,\rho(l)}$ be mutually disjoint disks embedded in S^3. Suppose that they satisfy the following conditions;

(1) $\psi_i(B^3) \cap f_1(\theta) = \emptyset$ for each i,
(2) $b_{i,k} \cap f_1(\theta) = \partial b_{i,k} \cap (f_1(\theta) - f(v_1 \cup v_2))$ is an arc for each i, k,

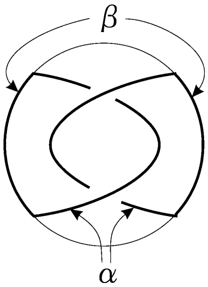

Fig. 2.1.

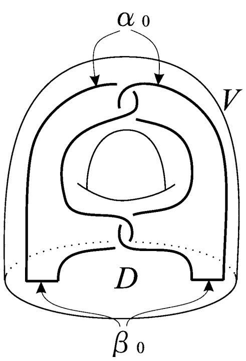

Fig. 2.2.

(3) $b_{i,k} \cap (\bigcup_{j=1}^{l} \psi_j(B^3)) = \partial b_{i,k} \cap \psi_i(B^3)$ is a component of $\psi_i(\beta_i)$ for each i, k,
(4) $(\bigcup_{k=1}^{\rho(i)} b_{i,k}) \cap \psi_i(B^3) = \psi_i(\beta_i)$ for each i.

Let f_2 be a spatial θ-curve defined by

$$f_2(\theta) = f_1(\theta) \cup \left(\bigcup_{i,k} \partial b_{i,k}\right) \cup \left(\bigcup_{i=1}^{l} \psi_i(\alpha_i)\right) - \bigcup_{i,k} \operatorname{int}\bigl(\partial b_{i,k} \cap f_1(\theta)\bigr) - \bigcup_{i=1}^{l} \psi_i(\operatorname{int}\beta_i),$$

where the labels of $f_2(\theta)$ coincides that of $f_1(\theta)$ on $f_1(\theta) - \bigcup_{i,k} b_{i,k}$. When (α_i, β_i) is a C_k-link model, we call $\psi_i(B^3)$ a *C_k-link ball*. We set $\mathcal{B}_i = ((\alpha_i, \beta_i), \psi_i, \{b_{i,1}, \ldots, b_{i,\rho(i)}\})$ and call $\mathcal{B}_i$ a *C_k-chord* when (α_i, β_i) is a C_k-link model. We denote f_2 by $\Omega(f_1; \{\mathcal{B}_1, \ldots, \mathcal{B}_l\})$ and call it a *band description* of f_2. We also say f_2 is a *band sum* of f_1 and link models $(\alpha_1, \beta_1), \ldots, (\alpha_l, \beta_l)$.

By the arguments similar to that in proof of Lemma 3.6 [20], we have

Lemma 2.1. *Two spatial θ-curves f_1 and f_2 are C_k-equivalence if and only if there are spatial θ-curves f_1' and f_2' such that f_i' is ambient isotopic to f_i $(i = 1, 2)$ and f_2' is a band sum of f_1' and some C_k-link models.*

In the following lemma, the former assertion follows directly from Lemma 3.9 in [20] and the latter can be shown by the similar arguments as in proof of Lemma 3.9 in [20].

Lemma 2.2. *A local move as illustrated in Fig.* 2.3 (*respectively Fig.* 2.4) *is realized by a C_{j+k}-move* (*respectively C_{j+k+1}-move*).

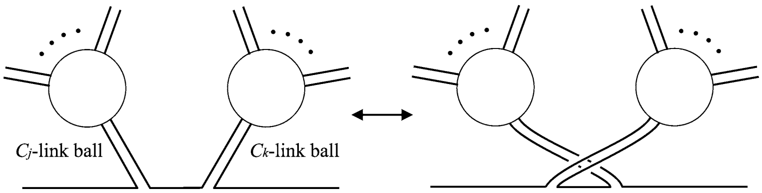

Fig. 2.3.

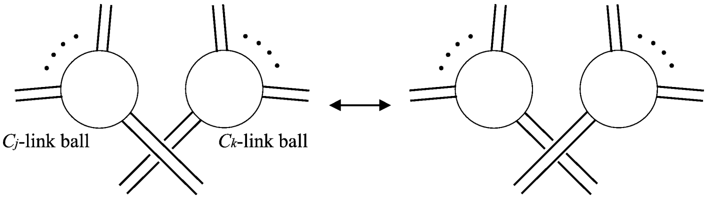

Fig. 2.4.

Proof of Theorem 1.5. (1) Suppose $f \in [f_0]_k$, then by Lemma 2.1, we may assume that f is a band sum of f_0 and some C_k-link models. Let g be a spatial θ-curve. Since $\Gamma = [f_0]_2$, we may suppose g is a band sum of f_0 and some C_2-link models. It is not hard to see that $f\#g$ and $g\#f$ are transposed each other by the moves as in Figs. 2.3 and 2.4, where we consider the case $j = 2$, and ambient isotopies. Thus by Lemma 2.2 we have $f\#g$ is C_{k+2}-equivalent to $g\#f$. Hence we have $[f]_{k+2}[g]_{k+2} = [f\#g]_{k+2} = [g\#f]_{k+2} = [g]_{k+2}[f]_{k+2}$.

(2) Suppose that both $[f_1]_k$ and $[f_2]_k$ belong to H_k^l. Then we note that $[f_1]_k[f_2]_k = [f_1\#f_2]_k \in H_k^l$. If $[f]_k$ belongs to H_k^l, then by Lemma 2.1, we may assume that $f = \Omega(f_0; \{\mathcal{B}_1, \dots, \mathcal{B}_n\})$ and $f_0 = \Omega(f; \{\mathcal{B}'_1, \dots, \mathcal{B}'_m\})$ for some C_l-chords $\mathcal{B}_1, \dots, \mathcal{B}_n$, $\mathcal{B}'_1, \dots, \mathcal{B}'_m$. By using Sublemma 3.5 in [20] repeatedly, there are C_l-chords $\mathcal{B}''_1, \dots, \mathcal{B}''_m$ such that f_0 is ambient isotopic to $\Omega(f_0; \{\mathcal{B}_1, \dots, \mathcal{B}_n\mathcal{B}''_1, \dots, \mathcal{B}''_m\})$. By Lemma 2.2, we can deform $\Omega(f_0; \{\mathcal{B}_1, \dots, \mathcal{B}_n\mathcal{B}''_1, \dots, \mathcal{B}''_m\})$ into $\Omega(f_0; \{\mathcal{B}_1, \dots, \mathcal{B}_n\})\#\Omega(f_0; \{\mathcal{B}''_1, \dots, \mathcal{B}''_m\})$ by C_{2l}-moves and ambient isotopies, i.e., f_0 and $f\#\Omega(f_0; \{\mathcal{B}''_1, \dots, \mathcal{B}''_m\})$ are C_{2l}-equivalent. Since $2l \geqslant k$, f_0 and $f\#\Omega(f_0; \{\mathcal{B}''_1, \dots, \mathcal{B}''_m\})$ are C_k-equivalent. This implies that $[\Omega(f_0; \{\mathcal{B}''_1, \dots, \mathcal{B}''_m\})]_k = [f]_k^{-1} \in H_k^l$. Therefore H_k^l is a subgroup of G_k. By the arguments similar to that in (1), we see that H_k^l is abelian. □

Let v be an invariant of the embeddings of a graph that takes values in an abelian group. We call v a *Vassiliev invariant of type* $(k_1, \dots, k_l)$ if, for any singular spatial graph $\{f_P \mid P \subset \{1, \dots, l\}\}$ of type $(k_1, \dots, k_l)$,

$$\sum_{P \subset \{1,\dots,l\}} (-1)^{|P|} v(f_P) = 0.$$

Proof of Theorem 1.7. The 'only if' part follows from Theorem 1.1(1). We shall show 'if' part. Let $\varphi : \Gamma \to H_{2k}^k$ be a map defined as follows:

$$\varphi(f) = \begin{cases} [f]_{2k} & \text{if } f \in [f_0]_k, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, φ is an invariant. Now we will show that φ is a Vassiliev invariant of type (k, k). Let $\{h_P \mid P \subset \{1, 2\}\}$ be a singular spatial θ-curve of type (k, k). Since $\{h_P \mid P \subset \{1, 2\}\} \subset [h_\emptyset]_k$, we have $\varphi(h_P) = 0$ if $[h_\emptyset]_k \neq [f_0]_k$. So we may suppose that $[h_\emptyset]_k = [f_0]_k$. Then we have

$$\sum_{P \subset \{1,2\}} (-1)^{|P|} \varphi(h_P) = [h_\emptyset]_{2k} - [h_{\{1\}}]_{2k} - [h_{\{2\}}]_{2k} + [h_{\{1,2\}}]_{2k}.$$

Since $[h_\emptyset]_k = [f_0]_k$, by Lemma 2.1, we may assume that $h_\emptyset = \Omega(f_0; \{\mathcal{B}_1, \dots, \mathcal{B}_n\})$ for some C_k-chords $\mathcal{B}_1, \dots, \mathcal{B}_n$. By Sublemma 3.1 in [20] (or Lemma 3.7 in the next section), there are C_k-chords $\mathcal{B}'_1, \mathcal{B}'_2$ such that $\Omega(h_\emptyset; \bigcup_{i \in P} \{\mathcal{B}'_i\})$ is ambient isotopic to h_P. By the arguments similar to that in the proof of Sublemma 3.5 in [20], we see that there are C_k-chords $\mathcal{B}''_1, \mathcal{B}''_2$ such that $\Omega(f_0; \{\mathcal{B}_1, \dots, \mathcal{B}_n\} \cup (\bigcup_{i \in P} \{\mathcal{B}''_i\}))$ is ambient isotopic to h_P. Since $\mathcal{B}_1, \dots, \mathcal{B}_n, \mathcal{B}''_1, \mathcal{B}''_2$ are C_k-chords, by Lemma 2.2, we have

$$\begin{aligned} &\bigl[\Omega\bigl(f_0; \{\mathcal{B}_1, \dots, \mathcal{B}_n\} \cup \{\mathcal{B}''_1, \mathcal{B}''_2\}\bigr)\bigr]_{2k} \\ &\quad = \bigl[\Omega\bigl(f_0; \{\mathcal{B}_1, \dots, \mathcal{B}_n\}\bigr)\#\Omega\bigl(f_0; \{\mathcal{B}''_1\}\bigr)\#\Omega\bigl(f_0; \{\mathcal{B}''_2\}\bigr)\bigr]_{2k}, \end{aligned}$$

and

$$\begin{aligned}&\big[\Omega\big(f_0;\{\mathcal{B}_1,\dots,\mathcal{B}_n\}\cup\{\mathcal{B}_i''\}\big)\big]_{2k}\\&\quad=\big[\Omega\big(f_0;\{\mathcal{B}_1,\dots,\mathcal{B}_n\}\big)\#\Omega\big(f_0;\{\mathcal{B}_i''\}\big)\big]_{2k}\quad (i=1,2).\end{aligned}$$

Hence we have

$$\begin{aligned}\sum_{P\subset\{1,2\}}(-1)^{|P|}\varphi(h_P) &= [f_\emptyset]_{2k}-\big[f_\emptyset\#\Omega\big(f_0;\{\mathcal{B}_1''\}\big)\big]_{2k}-\big[f_\emptyset\#\Omega\big(f_0;\{\mathcal{B}_2''\}\big)\big]_{2k}\\&\quad+\big[f_\emptyset\#\Omega\big(f_0;\{\mathcal{B}_1''\}\big)\#\Omega\big(f_0;\{\mathcal{B}_2''\}\big)\big]_{2k}\\&= 0\in H_{2k}^k.\end{aligned}$$

Therefore φ is a Vassiliev invariant of type (k,k). This and the assumption $v_{(k,k)}(f_1)=v_{(k,k)}(f_2)$ imply $\varphi(f_1)=\varphi(f_2)$. By the definition of φ, we have $[f_1]_{2k}=[f_2]_{2k}$. □

3. Disk/band surfaces and Vassiliev invariants of spatial graphs

A graph G is *trivalent* if the valence of any vertex of G is equal to 3. A graph G is *planar* if there exists an embedding $f_0:G\to\mathbb{R}^2$. A connected, planar graph G is said to be *prime* if, for any embedding $f_0:G\to\mathbb{R}^2$, there exist no simple closed curves C in $\mathbb{R}^2$ satisfying either the following (1) or (2) (cf. [17,6]), where A, B are the two components of $\mathbb{R}^2-C$.

(1) C meets $f_0(G)$ in a single point such that both $A\cap f_0(G)$ and $B\cap f_0(G)$ are non-empty.
(2) C meets $f_0(G)$ in two points such that both $A\cap f_0(G)$, $B\cap f_0(G)$ are neither empty nor single open arcs.

For any connected, planar graph G, we fix a planar embedding $f_0:G\to\mathbb{R}^2$ arbitrarily. The image $f_0(G)$ has complementary domains $D_1,D_2,\dots,D_n$ that are bounded and one unbounded D_0. The preimage $c_i=f_0^{-1}(\partial D_i)$ is a 1-complex which can be viewed as a 1-cycle in $H_1(G;\mathbb{Z})$. We call c_i $(i\neq 0)$, c_0 respectively a *boundary cycle* and the *outermost cycle* in G with respect to f_0.

For a spatial embedding $f:G\to S^3$ of a graph G, a *disk/band surface* S of $f(G)$ is a compact, orientable surface in S^3 such that $f(G)$ is a deformation retract of S contained in $\operatorname{int} S$ [6].

In [13], Soma, Sugai and the author showed the following theorem.

Theorem 3.1 [13, Theorem 1]. *Suppose that G is a connected, planar, prime and trivalent graph, and $f_0:G\to\mathbb{R}^2$ is an embedding. Then, for any embedding $f:G\to S^3$, there exists the unique disk/band surface S of $f(G)$ up to ambient isotopy of which the Seifert pairings satisfying the following equation.*

$$\langle f(c_i), f(c_j)\rangle_S = \begin{cases} -\mathrm{lk}\big(f(c_i), f(c_0)\big) - \sum\limits_{c_i\cap c_k=\emptyset} \mathrm{lk}\big(f(c_i), f(c_k)\big) & \text{if } i=j \text{ and } c_i\cap c_0=\emptyset,\\ 0 & \text{if } i\neq j \text{ and } c_i\cap c_j\neq\emptyset,\\ 0 & \text{if } i=j \text{ and } c_i\cap c_0\neq\emptyset,\\ \mathrm{lk}\big(f(c_i), f(c_j)\big) & \text{if } c_i\cap c_j=\emptyset, \end{cases}$$

where c_i, c_j, c_k are boundary cycles and c_0 is the outermost cycle with respect to f_0.

We call the disk/band surface above the *canonical disk/band surface* for f. Note that the Seifert linking form of the canonical disk/band surface depends only on the linking numbers of pairs of disjoint cycles. If G is the θ-curve or the complete graph with 4 vertices, then the canonical disk/band surface is same as the *disk/band surface with zero Seifert linking form* that is defined in [6]. By the proof of Theorem 1 in [13], we note that the canonical disk/band surface is given as the image of an embedding of the regular neighborhood S_0 of $f_0(G)$ in $\mathbb{R}^2$. Thus by fixing orientation and label of ∂S_0, we have an ordered, oriented link as the image of an embedding of ∂S_0. From now on, we always assume that, for each graph G, ∂S_0 has fixed orientation and label.

Let (T_1, T_2) be a local move, $t_{11}, \ldots, t_{1n}$ the components of T_1 and $t_{21}, \ldots, t_{2n}$ the components of T_2 with $\partial t_{1i} = \partial t_{2i}$ $(i = 1, \ldots, n)$. Let N_{1i} and N_{2i} be regular neighbourhoods of t_{1i} and t_{2i} in B^3 respectively such that $N_{1i} \cap \partial B^3 = N_{2i} \cap \partial B^3$ $(i = 1, \ldots, n)$ and $N_{1i} \cap N_{1j} = N_{2i} \cap N_{2j} = \emptyset$ $(1 \leqslant i < j \leqslant n)$. Let α_i $(i = 1, \ldots, n)$ be disjoint union of properly embedded l_i arcs in $B^2 \times [0,1]$ as illustrated in Fig. 3.1. Let $\psi_{ji} : B^2 \times [0,1] \to N_{ji}$ be a homeomorphism with $\psi_{ji}(B^2 \times \{0,1\}) = N_{ji} \cap \partial B^3$ for $j = 1, 2,$ $i = 1, \ldots, n$. Suppose that $\psi_{1i}(\partial\alpha_i) = \psi_{2i}(\partial\alpha_i)$ and $\psi_{1i}(\alpha_i)$ and $\psi_{2i}(\alpha_i)$ are ambient isotopic in B^3 relative to ∂B^3. Then we say that a local move $(\bigcup_{i=1}^n \psi_{1i}(\alpha_i), \bigcup_{i=1}^n \psi_{2i}(\alpha_i))$ is a *parallel of* (T_1, T_2) *with weight* $(l_1, \ldots, l_n)$.

Proposition 3.2. *Let G be a connected, planar, prime and trivalent graph. Let $f_i : G \to S^3$ $(i = 1, 2)$ be embeddings and S_i the canonical disk/band surface for f_i $(i = 1, 2)$. If f_1 and f_2 are C_k-equivalent, then ∂S_1 and ∂S_2 are C_k-equivalent.*

Let T_1 and T_2 be tangles. We say that T_2 is *obtained from T_1 by a local move* (U_1, U_2) if there is an orientation preserving embedding $h : B^3 \to \mathrm{int}\, B^3$ such that $S_i \cap h(B^3) = h(U_i)$ for $i = 1, 2$ and $T_1 - h(B^3) = T_2 - h(B^3)$. Two tangles T_1 and T_2 are *C_k-equivalent* if T_2 is obtained from T_1 by a finite sequence of C_k-moves and ambient isotopies relative ∂B^3.

Lemma 3.3 (cf. [3, Claim on p. 26]). *Let (T_1, T_2) be a parallel of a C_k-move. Then T_1 and T_2 are C_k-equivalent.*

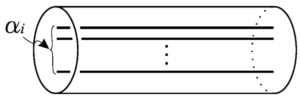

Fig. 3.1.

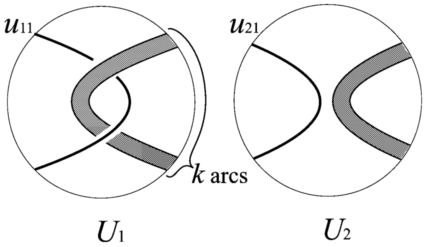

Fig. 3.2.

Proof. Let (U_1, U_2) be a C_k-move, $u_{11}, \dots, u_{1k+1}$ and $u_{21}, \dots, u_{2k+1}$ the components of U_1 and U_2 respectively. Suppose that (T_1, T_2) is a parallel of (U_1, U_2) with weight $(l_1, \dots, l_{k+1})$. We give a proof by induction on $l_1 \times \cdots \times l_{k+1}$. In the case that $l_1 \times \cdots \times l_{k+1} = 1$, it obviously holds. Suppose $l_1 \times \cdots \times l_{k+1} \geqslant 2$. We may suppose $l_1 \geqslant 2$. By Lemma 2.1 in [20], we may assume that the C_k-move (U_1, U_2) is as illustrated in Fig. 3.2, i.e., the arcs except for u_{i1} are contained in the shaded part in U_i $(i = 1, 2)$. It is not hard to see that T_2 is obtained from T_1 by l_1 local moves that are parallels of (U_1, U_2) with weight $(1, l_2, \dots, l_{k+1})$. This completes the proof. □

Proof of Proposition 3.2. In the case $k = 1$, it clearly holds. We consider the case $k \geqslant 2$. It is sufficient to consider the case that f_2 is obtained from f_1 by a single C_k-move. Suppose that f_2 is obtained from f_1 by a C_k-move. Then there is an embedding $h : B^3 \to S^3$ such that $(h^{-1}(f_1(\theta)), h^{-1}(f_2(\theta)))$ is a C_k-move and $f_1(\theta) - h(B^3) = f_2(\theta) - h(B^3)$ together with the labels. We may suppose that $(h^{-1}(\partial S_1), h^{-1}(\partial S_2))$ is a parallel of the C_k-move $(h^{-1}(f_1(\theta)), h^{-1}(f_2(\theta)))$ with weight $(2, \dots, 2)$. Then we have a new disk/band surface S_2' for f_2 from S_1 by the local move $(h^{-1}(\partial S_1), h^{-1}(\partial S_2))$. Since C_k-move $(n \geqslant 2)$ does not change the linking number, by Theorem 3.1, we have $\langle f_1(c_i), f_1(c_j)\rangle_{S_1} = \langle f_2(c_i), f_2(c_j)\rangle_{S_2}$, and since the local move $(h^{-1}(\partial S_1), h^{-1}(\partial S_2))$ does not change the Seifert linking form of a disk/band surface, we have $\langle f_1(c_i), f_1(c_j)\rangle_{S_1} = \langle f_2(c_i), f_2(c_j)\rangle_{S_2'}$. So we have $\langle f_2(c_i), f_2(c_j)\rangle_{S_2} = \langle f_2(c_i), f_2(c_j)\rangle_{S_2'}$. By Theorem 3.1, S_2 is ambient isotopic to S_2'. Thus ∂S_2 is obtained from ∂S_1 by a parallel of a C_k-move. Lemma 3.3 completes the proof. □

Let G be a connected, planar, prime and trivalent graph and $\Gamma(G)$ the set of spatial graph types. Let v be an invariant of ordered, oriented links that takes values in an abelian group A. Then we define a map $s : \Gamma(G) \to A$ as $s(f) = v(\partial S)$, where S is the canonical disk/band surface for f. By Theorem 3.1, s is an invariant of f. We call s the *invariant induced from* v.

Theorem 3.4. *Let G be a connected, planar, prime and trivalent graph and $\Gamma(G)$ the set of spatial graph types. Let v be a Vassiliev invariant of type $(k_1, \dots, k_l)$ for ordered, oriented links. Then the invariant for $\Gamma(G)$ induced from v is a Vassiliev invariant of type $(k_1, \dots, k_l)$.*

In Theorem 3.4, the case of that a graph is the θ-curve and $k_1 = \cdots = k_l = 1$ is given by Stanford [14].

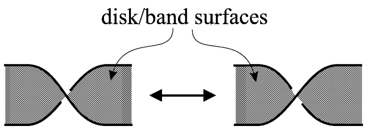

Fig. 3.3.

By the arguments similar to that in the proof of Lemma 1.4 in [14], we have

Lemma 3.5. *Let v be a Vassiliev invariant of type $(k_1, \dots, k_l)$ for ordered, oriented links and s the invariant induced from v. Let $\{f_P \mid P \subset \{1, \dots, l\}\}$ be a singular spatial graph of type $(k_1, \dots, k_l)$. Let L_P be the ordered, oriented link that is the boundary of the canonical disk/band surface for f_P $(P \subset \{1, \dots, l\})$. Suppose there are mutually disjoint embeddings $h_{ij} : B^3 \to S^3$ $(i = 1, \dots, l,\ j = 1, \dots, n_i)$ such that*

(1) $L_\emptyset - \bigcup_{i,j} h_{ij}(B^3) = L_P - \bigcup_{i,j} h_{ij}(B^3)$ *together with orientations and labels of the components for any subset $P \subset \{1, \dots, l\}$,*
(2) $(h_{ij}^{-1}(L_\emptyset), h_{ij}^{-1}(L_{\{1,\dots,l\}}))$ *is a C_{k_i}-move $(i = 1, \dots, l,\ j = 1, \dots, n_i)$, and*
(3) $$L_P \cap h_{ij}(B^3) = \begin{cases} L_{\{1,\dots,l\}} \cap h_{ij}(B^3) & \text{if } i \in P, \\ L_\emptyset \cap h_{ij}(B^3) & \text{otherwise.} \end{cases}$$

Then we have

$$\sum_{P \subset \{1,\dots,l\}} (-1)^{|P|} s(f_P) = \sum_{P \subset \{1,\dots,l\}} (-1)^{|P|} v(L_P) = 0.$$

The following lemma follows directly from the proof of Theorem 1 in [13].

Lemma 3.6. *Let G be a connected, planar, prime and trivalent graph, $f_0 : G \to \mathbb{R}^2$ an embedding, and S_0 the regular neighborhood of $f_0(G)$ in $\mathbb{R}^2$. Let S be a disk/band surface for an embedding f such that S is the image of an embedding of S_0 that is an extension of f. Then the canonical disk/band surface for f is obtained from S by a finite sequence of the moves as illustrated in Fig.* 3.3.

In the definition of band sum in Section 2, by replacing $f_i(\theta)$ with T_i $(i = 1, 2)$, we can define that T_2 is a *band sum* of T_1 and link models $(\alpha_1, \beta_1), \dots, (\alpha_l, \beta_l)$. By the arguments similar to that in proof of Lemma 3.6 [20], we have the following lemma.

Lemma 3.7. *Two tangles T_1 and T_2 are C_k-equivalent if and only if there are tangles T_1' and T_2' such that T_i' is ambient isotopic to T_i $(i = 1, 2)$ relative ∂B^3 and T_2' is a band sum of T_1' and some C_k-link models.*

Lemma 3.8. *Let T_1 and T_2 be tangles. If T_2 is obtained from T_1 by a parallel of a C_k-move, then there are tangles T_1', T_2' and mutually disjoint, orientation preserving embeddings $h_i : B^3 \to \operatorname{int} B^3$ $(i = 1, \dots, n)$ such that*

(1) *T'_j is ambient isotopic to T_j $(j = 1, 2)$ relative ∂B^3,*
(2) *$T'_1 - \bigcup_i h_i(B^3) = T'_2 - \bigcup_i h_i(B^3)$, and*
(3) *$(h_i^{-1}(T'_1), h_i^{-1}(T'_2))$ is a C_k-move $(i = 1, \ldots, n)$.*

Proof. By Lemma 3.3, T_1 and T_2 are C_k-equivalent. By Lemma 3.7, there are tangles T'_1 and T'_2 such that T'_j is ambient isotopic to T_j $(j = 1, 2)$ relative ∂B^3 and that T'_2 is a band sum of T'_1 and some C_k-link models $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$. Since $(\alpha_i, \hat{\beta}_i)$ $(i = 1, \ldots, n)$ are C_k-moves, we have the conclusion. □

Proof of Theorem 3.4. Since $\{f_P \mid P \subset \{1, \ldots, l\}\}$ is a singular spatial graph of type $(k_1, \ldots, k_l)$, by the definition, there are mutually disjoint, orientation preserving embeddings $h_i : B^3 \to S^3$ $(i = 1, \ldots, l)$ such that

(1) $f_\emptyset(G) - \bigcup_i h_i(B^3) = f_P(G) - \bigcup_i h_i(B^3)$ together with the labels for any subset $P \subset \{1, \ldots, l\}$,
(2) $(h_i^{-1}(f_\emptyset(G)), h_i^{-1}(f_{\{1,\ldots,l\}}(G)))$ is a C_{k_i}-move $(i = 1, \ldots, l)$, and
(3) $$f_P(G) \cap h_i\bigl(B^3\bigr) = \begin{cases} f_{\{1,\ldots,l\}}(G) \cap h_i(B^3) & \text{if } i \in P, \\ f_\emptyset(G) \cap h_i(B^3) & \text{otherwise.} \end{cases}$$

Let $S_\emptyset$ be the canonical disk/band surface for $f_\emptyset$. By considering the intersections $S_\emptyset \cap h_i(B^3)$ $(i = 1, \ldots, l)$, we find disk/band surfaces S_P for f_P $(P \subset \{1, \ldots, l\})$ such that

(1) $S_\emptyset - \bigcup_i h_i(B^3) = S_P - \bigcup_i h_i(B^3)$,
(2) $(h_i^{-1}(\partial S_\emptyset), h_i^{-1}(\partial S_{\{1,\ldots,l\}}))$ is a parallel of a C_{k_i}-move $(i = 1, \ldots, l)$, and
(3) $$S_P \cap h_i\bigl(B^3\bigr) = \begin{cases} S_{\{1,\ldots,l\}} \cap h_i(B^3) & \text{if } i \in P, \\ S_\emptyset \cap h_i(B^3) & \text{otherwise.} \end{cases}$$

By the proof of Proposition 3.2, if $k_i \geqslant 2$ for any $i \in P$, then S_P is the canonical disk/band surface. Set $h_i = h_{i1}$ $(i = 1, \ldots, l)$. By Lemma 3.6, there are mutually disjoint, orientation preserving embeddings $h_{ij} : B^3 \to S^3$ $(i = 1, \ldots, l,\ j = 1, \ldots, n_i)$, where $n_i = 1$ if $k_i \geqslant 2$, and the canonical disk/band surfaces S'_P for f_P such that

(1) $S'_\emptyset - \bigcup_{i,j} h_{ij}(B^3) = S'_P - \bigcup_{i,j} h_{ij}(B^3)$,
(2) $(h_{ij}^{-1}(\partial S'_\emptyset), h_{ij}^{-1}(\partial S'_{\{1,\ldots,l\}}))$ is a parallel of C_{k_i}-move $(i = 1, \ldots, l,\ j = 1, \ldots, n_i)$, and
(3) $$S'_P \cap h_{ij}\bigl(B^3\bigr) = \begin{cases} S'_{\{1,\ldots,l\}} \cap h_{ij}(B^3) & \text{if } i \in P, \\ S'_\emptyset \cap h_{ij}(B^3) & \text{otherwise.} \end{cases}$$

By combining this, Lemmas 3.8 and 3.5, we have the conclusion.

Let G be a connected, planar, prime and trivalent graph and $E(G) = \{e_1, \ldots, e_n\}$ the set of edges of G. Let S_f be the canonical disk/band surface for a spatial embedding f of G, and let $S_f(x_1, \ldots, x_n; y_1, \ldots, y_n)$ $(x_i \in \mathbb{Z}, y_j \in \{-1, 0, 1\})$ be a surface obtained from

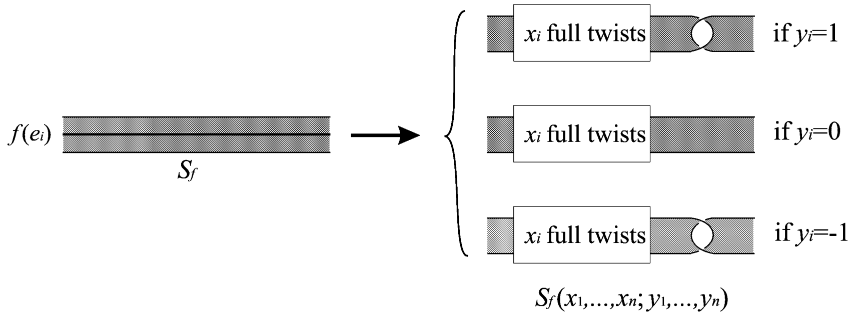

Fig. 3.4.

S_f as illustrated in Fig. 3.4. We note that $S_f(x_1, \ldots, x_n; y_1, \ldots, y_n)$ depends only on S_f and the integers $x_1, \ldots, x_n, y_1, \ldots, y_n$. This means $S_f(x_1, \ldots, x_n; y_1, \ldots, y_n)$ is the unique surface for f. Let v be an invariant of ordered, oriented links that takes values in an abelian group A. Then we can define an invariant

$$s_{(x_1,\ldots,x_n;y_1,\ldots,y_n)} : \Gamma(G) \to A$$

as $s_{(x_1,\ldots,x_n;y_1,\ldots,y_n)}(f) = v(\partial S_f(x_1, \ldots, x_n; y_1, \ldots, y_n))$. We call $s_{(x_1,\ldots,x_n;y_1,\ldots,y_n)}$ the *invariant induced from* v with respect to $x_1, \ldots, x_n, y_1, \ldots, y_n$. By the arguments similar to that in the proofs of Proposition 3.2 and Theorem 3.4, we have the following theorem.

Theorem 3.9. *Let G be a connected, planar, prime and trivalent graph and $E(G) = \{e_1, \ldots, e_n\}$ the set of edges of G. Then the followings hold.*

(1) *Let f_1 and f_2 be spatial graphs and $S_i(x_1, \ldots, x_n; y_1, \ldots, y_n)$ the surface obtained from the canonical disk/band surface for f_i $(i = 1, 2)$. If f_1 and f_2 are C_k-equivalent, then $\partial S_1(x_1, \ldots, x_n; y_1, \ldots, y_n)$ and $\partial S_2(x_1, \ldots, x_n; y_1, \ldots, y_n)$ are C_k-equivalent.*

(2) *Let v be a Vassiliev invariant of type $(k_1, \ldots, k_l)$ for ordered, oriented links. Then the invariant $s_{(x_1,\ldots,x_n;y_1,\ldots,y_n)}$ for spatial embeddings of G induced from v is a Vassiliev invariant of type $(k_1, \ldots, k_l)$.*

Proof of Theorem 1.8. Suppose that G_k is abelian. Let f_1 and f_2 be spatial θ-curves as illustrated in Fig. 3.5. Since G_k is abelian, $g = f_1 \# f_1 \# f_2$ and $h = f_1 \# f_2 \# f_1$ are C_k-equivalent. Then, by Theorem 1.1(1),

$$g - h \in \mathcal{V}(\underbrace{1, \ldots, 1}_{k}).$$

This means that g and h cannot be distinguished by any Vassiliev invariant of order $\leqslant k-1$. Let $S_g(1, 1, -1; 0, 0, 1)$ and $S_h(1, 1, -1; 0, 0, 1)$ be the surfaces obtained from the canonical disk/band surfaces for g and h respectively. By deleting components corresponding to a cycle $e_1 \cup e_2$ from $\partial S_g(1, 1, -1; 0, 0, 1)$ and $\partial S_h(1, 1, -1; 0, 0, 1)$, we obtain pretzel knots $K_g = P(3, 3, -3, -2)$ and $K_h = P(3, -3, 3, -2)$ respectively, see Fig. 3.6. Let v be a Vassiliev invariant for oriented link of order $\leqslant k-1$. By combining Theorem 3.9(1) and the fact that a C_k-moves preserves Vassiliev invariants of order $\leqslant k-1$ [3] (or simply by Theorem 3.9(2)), we have $v(\partial S_g(1, 1, -1; 0, 0, 1)) = v(\partial S_h(1, 1, -1; 0, 0, 1))$. Hence

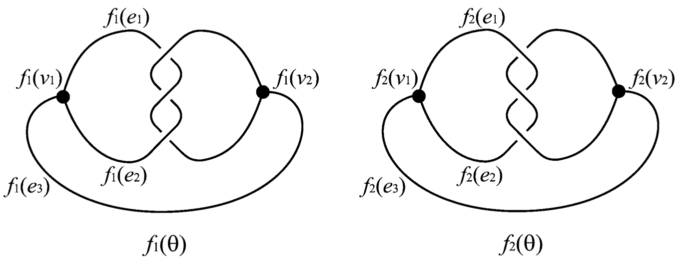

Fig. 3.5.

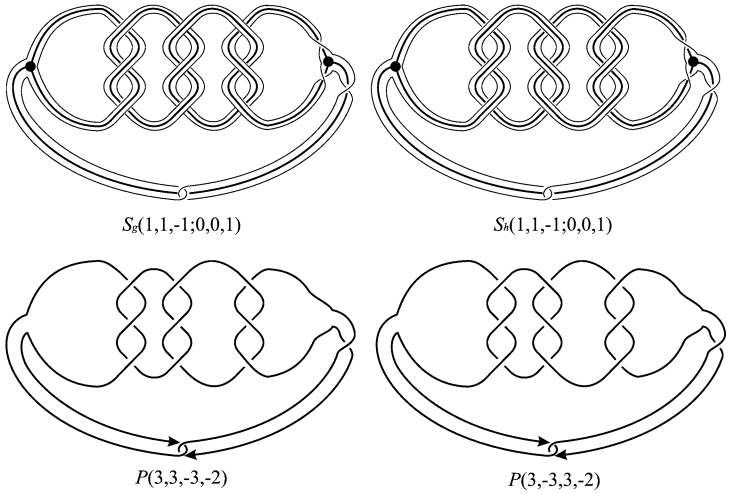

Fig. 3.6.

$v(K_g) = v(K_h)$. Let $X_K^{(3)}(q, q^3)^*$ be the quantum invariant of a knot K corresponding to the representation of the partition $(2, 1)$ of the quantum enveloping algebra $\mathcal{U}_q(sl_4)$. (Fuller details about $X_K^{(3)}(q, q^3)^*$ can be seen in [12].) The computer software 'K2K' [4], whose methods of the calculation is based on Murakami's one [10], gives us

$$\begin{aligned}
&\frac{X_{K_g}(q^2, q^6)^* - X_{K_h}(q^2, q^6)^*}{X_O(q^2, q^6)^*} \\
&\quad = q^s(-1+q)^{11}(1+q)^{11}\big(1+q^2\big)^3\big(1-q+q^2\big)^3\big(1+q+q^2\big)^3 \\
&\qquad \times \big(1+q^4\big)^2\big(1-q^2+q^4\big)\big(1-q+q^2-q^3+q^4-q^5+q^6\big)^2 \\
&\qquad \times \big(1+q+q^2+q^3+q^4+q^5+q^6\big)^2\big(1+q^8\big) \\
&\qquad \times \big(1-q^2+q^4-q^6+q^8-q^{10}+q^{12}\big),
\end{aligned}$$

where s is some integer and O is a trivial knot. Since this is divisible by $(1-q)^{11}$ and is not divisible by $(1-q)^{12}$, these pretzel knots can be distinguished by a Vassiliev invariant of order $\leqslant 11$ [1]. Hence we have $k < 12$. This completes the proof. □

Remark. Our arguments in the proof above for distinguishing two mutant knots K_g and K_h can be found in [11]. In which, Murakami showed that the Conway's 11-crossing knot and the Kinoshita–Terasaka knot can be distinguished by a Vassiliev invariant of order 11, and that any Vassiliev invariant of order $\leqslant 10$ cannot distinguish mutant pairs of knots.

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